CHAPTER IX

Polynomial equations with matrix roots

9.1. Introduction.

For algebras where the multiplicative structure is noncommutative, polynomial arguments need enhancement. For the case where we have matrix solutions to polynomial equations with complex coefficients, the use of companion matrices and the Cayley-Hamilton theorem provides at least one, but non-unique, solution to such a polynomial equation. The general noncommutative case for matrix polynomials with matrix coefficients is more extensive than that of mere quaternions, being available in the hyperintricate methodology for general matrix rings, the matrices of which may be singular. In this chapter we investigate how polynomial theory transforms under the extension to these algebras.

Intricate and hyperintricate numbers were designed to investigate what happens to Galois theory for noncommutative algebras. Automorphism theory is described in the next chapter.

For intricate and hyperintricate polynomials unique factorisation is invalid in general. We provide an example, showing that there exist at least three distinct intricate factorisations of the polynomial equation $x^2 – a^2 = 0$. There are more.

We find for polynomials in x that are not described by multifunctions, that is, differing values of x are not used in the same polynomial, that any constraints in the commutative case carry over to the noncommutative one, provided coefficients are abelian, using 'J-abelian' algebra for hyperintricate numbers. A general hyperintricate polynomial with intricate coefficients has solvability constraints more stringent than the J-abelian case.

We obtain detailed new types of solution for solvable polynomial equations in degree less than 5 extended to intricate variables where the coefficients commute with the variables. We discuss solutions by equating hyperintricate parts and the relaxation conditions arising from and necessary for such methods in order to obtain solutions.

9.2. Do solvability criterions for fields extend to matrix algebras?

Note that a polynomial equation in complex variables and coefficients if written as

$$x^n + Tx^{n-1} + Ux^{n-2} + \ldots + W = 0, \qquad (1)$$

reduces to the equations

$$A + B + \ldots + D = T \qquad (2)$$
$$AB + AC + ... + BA + BC + \ldots = U$$
$$ABC\ldots D = W,$$

which are invariant under permutations of A and B, A and C, B and C etc., that is, of n objects. Galois theory, investigated and challenged from chapter X onwards, states there is no equation to convert A, B, C etc. in terms of combinations of T, U, … W for n > 4.

Anyway, the argument for reducing considerations to groups of permutations breaks down in the more complicated situation of

$$AB \neq BA$$

etc. – noncommutative algebras. If we expand solutions to belong to these algebras, we can represent them by hyperintricate numbers.

9.3. Intricate zeros.

As is proved in chapter I, the number of integer intricate factorisations of any integer, including zero, is countably infinite. If we multiply two intricate numbers and set them to zero

$$(a1 + bi + c\alpha + d\phi)(p1 + qi + r\alpha + s\phi) = 0, \qquad (1)$$

then on equating intricate parts to zero and eliminating a in two ways

$$b(p^2 + q^2 - r^2 - s^2) = 0.$$

A similar argument can be made on eliminating p.

The expression $(p^2 + q^2 - r^2 - s^2)$ is the determinant of its intricate number, so we are dealing with a singular matrix; in other words, its intricate inverse does not exist. This may be reformulated by taking the determinants of both sides of (1) and noting that the determinant of a product is the product of the determinants. □

We can therefore identify the type of matrix polynomial for which we are obtaining zeros as
Definition 9.3.1. A *solo zero expression* is one in which a solution may be obtained for each root separately, and
Definition 9.3.2. A *composite zero expression* is one where it is possible that no individual term is zero, but the composite expression is zero.

On taking the determinant of a composite zero polynomial of degree n

$$\prod_{k=1}^{n}(a_k 1 + b_k i + c_k \alpha + d_k \phi) = 0, \qquad (2)$$

since det 0 = 0 and the determinant of a product is the product of the determinants, we see that the determinant of at least one factor must be zero, but for a solo zero expression this includes the fact that the determinants of all factors at their roots are zero. □

9.4. Non-unique factorisation for hyperintricate polynomials.

The equaliser of two polynomials f(X) and g(X) is their intersection, so a solo zero of a polynomial equation is an equaliser of f(X) with the constant zero polynomial.

We note that hyperimaginary, hyperactual and hyperphantom polynomials are commutative.

Theorem 9.4.1 (failure of the fundamental theorem of algebra for matrix polynomials). *Unique factorisation fails for solo zero hyperintricate polynomials of degree* > 1.

Proof. We give examples of this. Let a be a real number. Both of the polynomial equations

$$(X - a)(X + a) = X^2 - a^2 = 0 \qquad (1)$$

and

$$(X - a\alpha)(X + a\alpha) = X^2 + aX\alpha - a\alpha X - a^2 = 0, \qquad (2)$$

the latter treated as a commutative hyperactual polynomial equating to

$$X^2 - a^2 = 0,$$

amount to the same equation with non-unique solutions as intricates.

A third solution is

$$(X - a\phi)(X + a\phi) = 0. \; □ \qquad (3)$$

A case of a composite zero polynomial nearer to that of a solo zero polynomial occurs when the determinant of each term with a root is zero at that root. Then if we choose x = a = p in equation 9.3.(1) we obtain

$$(x^2 + b^2 - c^2 - d^2)(x^2 + q^2 - r^2 - s^2) = 0, \quad (4)$$

which can hold for various distinct a, b, c, d, q, r and s. ☐

9.5. Additive and multiplicative format.

The matrices we have been considering are square, with equal numbers of rows and columns. We will compare a matrix polynomial in left-additive format with one in multiplicative format. If X is a square matrix with i and t ∊ $\mathbb{N}_{\cup 0}$, where X^t is the product of t matrices X, and T_i are matrix coefficients with $T_n = 1$, then

$$\sum_{i=0}^{n} T_iX^i = T_0 + T_1X + ... + T_nX^n = 0 \quad (1)$$

is a matrix polynomial in left-additive format.

In multiplicative format, our polynomials have been of the form

$$\prod_{i=1}^{n}(X + S_i) = 0, \quad (2)$$

so that for n = 2 and more extensively

$$(X + S_1)(X + S_2) = X^2 + S_1X + XS_2 + S_1S_2 = 0, \quad (3)$$

and the coefficient in XS_2 is not in general in left-additive format.

If the coefficients S_i are real, then the equivalence of multiplicative and left- (or right-) additive format can be seen.

When X is intricate and the coefficients S_i are intricate, the conversion to left-additive format is made by the result of 1.9 of chapter I. When X is hyperintricate and the coefficients S_i are hyperintricate of the same dimension, then the corresponding result is found by applying this result to each layer. ☐

9.6. Equating hyperintricate parts.

In the allocation of a coefficient, c, to a hyperintricate basis element, say A_B, it is natural to consider c as a real number. However, for polynomial equations in a J-abelian variable as a summation of distinct such terms

$$X_Y = \Sigma cA_B$$

to a power n,

$$(X_Y)^n + E_F(X_Y)^{n-1} + ... + G_H = 0,$$

the imposition of the constraint

$$cA_B = 0 \text{ implies } c = 0,$$

which we call equating hyperintricate parts to zero, may result in equations of the type

$$(c^2 + d^2)A_B = 0 \text{ implies } (c^2 + d^2) = 0, \quad (1)$$

which are not satisfied by real c or d except both c = 0 and d = 0.

Classical solutions to polynomial equations, obtained by killing central terms in a linear substitution of variables to form a cyclotomic equation, may violate such assumptions.

We are then forced to allow

$$X_Y = \Sigma cA_B$$

in which c is a general intricate or hyperintricate variable. Thus the constraint of equating hyperintricate parts in order to obtain a solution may have to be relaxed if a solution by radicals is to be obtained.

We can bypass such considerations by appending a further index to X_Y and A_B of 1, and employ an exterior (more generally a relative) coefficient algebra by multiplying an intricate c by this layer of 1. Then in the first instance c is commutative with $A_{B,1}$, unless c is multiplied by other intricate coefficients.

As will become apparent in the next chapter, equating real and complex parts is related to the automorphism approach to solving polynomial equations. Equating hyperintricate parts corresponds to the generalisation of the automorphism method to matrix polynomials.

9.7. Intricate roots of unity.

Denote ω_n as a complex n-th root of unity, and Ω_n as an intricate such root. We will suppose that n = 2m + s, with m, n, k and s natural numbers. Let int be the integer part of a real number and 0! = 1. Then

$$\Omega_n{}^n = 1 = [a1 + bi + c\alpha + d\phi]^n,$$
$$= [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)]^m\Omega_n{}^s. \qquad (1)$$

This gives

$$\Omega_n{}^n = \sum(k = 0 \text{ to } m)[m!/(m - k)!k!]$$
$$\{2^k a^k(a^2 - b^2 + c^2 + d^2)^{m-k}(-b^2 + c^2 + d^2)^{\mathrm{int}(k/2)}(bi + c\alpha + d\phi)^{k - 2\mathrm{int}(k/2)}\}\Omega_n{}^s.$$

What is Ω_2?

$$\Omega_2{}^2 = 1 = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)].$$

Equating intricate parts, there are two alternatives, a = 0 or b = c = d = 0, where a = 0 implies

$$1 = -b^2 + c^2 + d^2,$$
$$\Omega_2 = \pm\sqrt{(-1 + c^2 + d^2)}i + c\alpha + d\phi.$$

For an extension to an interior algebra, the square root in parentheses can be imaginary. The alternative b = c = d = 0 results in

$$\Omega_2 = \pm 1 = a.$$

More extensively, if we wished to find the square root of $(a1 + bi + c\alpha + d\phi)$, say this is $(p1 + qi + r\alpha + s\phi)$, then

$$p^2 - q^2 + r^2 + s^2 + 2p(qi + r\alpha + s\phi) = a1 + bi + c\alpha + d\phi$$

so that when p = 0 this implies b = c = d = 0, and otherwise on equating intricate parts and solving a quadratic (we will assume restrictively that p is complex. Real coefficients may be generalised as those in an interior or exterior intricate algebra.)

$$p = \pm\sqrt{[2a \pm\sqrt{(4a^2 + b^2 - c^2 - d^2)}]},$$
$$q = b/2p,\ r = c/2p \text{ and } s = d/2p.$$

We will select and define Ω_3 by the following method

$$\Omega_3{}^3 = 1 = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)](a1 + bi + c\alpha + d\phi).$$
$$= a^3 + 3a(-b^2 + c^2 + d^2) + (3a^2 - b^2 + c^2 + d^2)(bi + c\alpha + d\phi).$$

Initially if we equate intricate parts, then

$$0 = 3a^2 - b^2 + c^2 + d^2$$

so

$$1 = -8a^3 = (-2a)^3.$$

If we now deviate from the allocation of a, b, c and d as real, but maintain the above relations, then for complex values of a:

$$-2a = 1,\ \omega_3 \text{ or } \omega_3^2,$$

and

$$-b^2 + c^2 + d^2 = -3a^2,$$
$$= -\tfrac{3}{4},\ -\tfrac{3}{4}\omega_3^2 \text{ or } -\tfrac{3}{4}\omega_3.$$

Hence the intricate cube roots of unity are

$$\Omega_3 = -\tfrac{1}{2} + (\tfrac{3}{4} + c^2 + d^2)^{\frac{1}{2}}i + c\alpha + d\phi,$$
$$\Omega_3 = -\tfrac{1}{2}\omega_3 + (\tfrac{3}{4}\omega_3^2 + c^2 + d^2)^{\frac{1}{2}}i + c\alpha + d\phi$$

or

$$\Omega_3 = -\tfrac{1}{2}\omega_3^2 + (\tfrac{3}{4}\omega_3 + c^2 + d^2)^{\frac{1}{2}}i + c\alpha + d\phi.\square$$

Alternatively, we can use the Euler relations of chapter XII, in which

$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta$$

and

$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta$$

imply in the first case that

$$1 = \cos(2\pi k) + i\sin(2\pi k)$$

and thus the complex nth root of unity is uniquely

$$\omega_n = 1^{1/n} = e^{2\pi ki/n}.$$

For intricate numbers, note that

$$e^{fi + g\alpha + h\phi} \neq e^{fi}e^{g\alpha}e^{h\phi},$$

since the exponentiated sum on the left commutes, but by the Euler relations, the product on the right does not.

However, if we consider

$$e^{(bi + c\alpha + d\phi)U},$$

where the square of $bi + c\alpha + d\phi$ satisfies $-b^2 + c^2 + d^2 = -1$, then a Taylor series expansion equates this to

$$\cos U + (bi + c\alpha + d\phi)\sin U.$$

Thus the intricate nth root of unity is

$$\Omega_n = 1^{1/n} = e^{2\pi k(bi + c\alpha + d\phi)/n},$$

where $-b^2 + c^2 + d^2 = -1$.

Non-unique values of Ω_n may satisfy, say

$$1 + \Omega_{3A} + \Omega_{3B}^2 \neq 0,$$

since putting

$$\Omega_{3A} = e\uparrow\{[2\pi k(\pm_A \sqrt{(1 + c^2 + d^2)}i + c\alpha + d\phi]/3\}$$

and

$$\Omega_{3B} = e\uparrow\{[2\pi k(\pm_B \sqrt{(1 + g^2 + h^2)}i + g\alpha + h\phi]/3\}$$

gives

$$1 + \Omega_{3A} + \Omega_{3B}^2 = \{[\pm_A \sqrt{(1 + c^2 + d^2)} \pm_B \sqrt{(1 + g^2 + h^2)}]i$$
$$+ (c - g)\alpha + (d - h)\phi\}(\sqrt{3})/2.$$

Also in general, as may be computed

$(\Omega_{3A})(\Omega_{3B}{}^2) \neq 1.$

More extensively an intricate number may be represented by

$e^p e^{(bi + c\alpha + d\phi)U}$

so the nth root is, if $-b^2 + c^2 + d^2 = -1$, and not +1 or 0,

$e^{p/n}[\cos(U/n) + (bi + c\alpha + d\phi)\sin(U/n)]$. ☐

9.8. The consequences of multiplicative solutions.

A general matrix X in n-hyperintricate format can be represented by

$$X = x_1 1_{1\ ...\ 1} + \text{non-real n-hyperintricate terms}$$
$$= x_1 1_{1\ ...\ 1} + Y,$$

where x_1 is a real number. This means multiplicative format 9.5.(2) is expressed as

$$\prod_{i=1}^{n}(x_1 1_{1...1} + S'_i) = 0, \qquad (1)$$

in which S'_i is a n-hyperintricate matrix containing real and hyperintricate parts.

On expanding out these terms, we find

$$x_1{}^n + (\sum_{i=1}^{n} S'_i)x_1{}^{n-1} + ... + \prod_{i=1}^{n}(S'_i) + 0, \qquad (2)$$

and equating hyperintricate parts the real component is

$$x_1{}^n + T_1 x_1{}^{n-1} + ... + T_n = 0, \qquad (3)$$

which has the usual solvability constraints according to Galois theory, other terms being those of hyperintricate components, say,

$$U_1 x_1{}^{n-1} + ... + U_n = 0. \qquad (4)$$

The form (1) is consistent, but it means that expressed in additive format, (3) and (4) must for instance be mutually consistent, and this cannot be the case for all freely chosen T_r and U_r. ☐

9.9. The Cayley-Hamilton theorem. [MB79]

In the exercises for chapter II we saw a 4 × 4 *companion matrix* is defined as

$$C = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}. \qquad (1)$$

and showed that the determinant

$$\det(C - xI) = \det\begin{bmatrix} -x & 0 & 0 & -a_0 \\ 1 & -x & 0 & -a_1 \\ 0 & 1 & -x & -a_2 \\ 0 & 0 & 1 & -x - a_3 \end{bmatrix} \qquad (2)$$

is given by

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^4. \qquad (3)$$

This can be extended to consider a polynomial of degree n arising from an n × n companion matrix, on multiplying g(x) by $(-1)^n$. To do this we can expand out (2) in terms of the final column by the method itemised in the lemma which follows. ☐

Lemma 9.9.1. *For an* n × n *matrix B, let* $B^{(jk)}$ *be the matrix obtained from* B *by striking out row* j *and column* k. *If* D *is the square matrix*

$$D_{jk} = (-1)^{j+k}\det(B^{(jk)}), \tag{4}$$

then

$$D^TB = \det(B)I, \tag{5}$$

where D^T *is the transpose of* D.

Proof. On expanding out row j in the definition of a determinant in chapter II, section 8, formula (2), we obtain (5) up to the sign $(-1)^{j+k}$. To obtain the sign, interchange row j with each previous row until j is first, and then do the same for column k. These operations do not affect $\det(B^{(jk)})$, but they change the sign of det(B) and thus the sign of D_{jk} [(i – 1) + (j – 1)] = (i + j – 2) times. ☐

Theorem 9.9.2. (**Cayley-Hamilton**). *Every* n × n *matrix satisfies its own eigenvalue equation* (*called a characteristic equation*).

Proof. The following factorisation holds

$$A^i - x^iI = (A^{i-1} + xA^{i-2} + ... + x^{i-1}I)(A - xI), \tag{6}$$

which we will rewrite using the x-matrix $L^{(i)}$ (a sum of powers of x each with related matrix coefficients $L^{(i)}$) as

$$A^i - x^iI = L^{(i)}(A - xI), \qquad i = 0, 1, 2, ... \tag{7}$$

By lemma 9.9.1, putting B = A – xI in equation (5),

$$D^T(A - xI) = \det(A - xI)I. \tag{8}$$

If we expand out the eigenvalue equation of A as

$$\det(A - xI) = c_0 + c_1x + ... + c_nx^n, \tag{9}$$

in which $c_n = \pm 1$, we define this equation to be c(x). So we have

$$D^T(A - xI) = c(x). \tag{10}$$

Now consider c(A). Then on using (7) we get

$$\begin{aligned} c(A) &= \textstyle\sum_{i=0}^{n} c_iA^i \\ &= \textstyle\sum_{i=0}^{n} c_ix^iI + \sum_{i=0}^{n} L^{(i)}(A - xI), \end{aligned} \tag{11}$$

and if we use equation (10) in (11) we obtain

$$c(A) = D^T(A - xI) + \textstyle\sum_{i=0}^{n} L^{(i)}(A - xI), \tag{12}$$

which can be represented using a new x-matrix E with

$$c(A) = E(A - xI). \tag{13}$$

If E ≠ 0 it has some degree k in x, so we expand it out to get

$$c(A) = \left(E^{(0)} + E^{(0)}x + \cdots + E^{(k)}x^k\right)(A - xI). \tag{14}$$

with $E^{(k)} \neq 0$ a nonzero matrix of scalars. Multiplying out gives a nonzero term $-E^{(k)}x^{k+1}$ of degree (k + 1) in x, a contradiction, since the left-hand side c(A) is a matrix of constants of degree 0 in x. Thus, E = 0, which implies c(A) = 0. ☐

On putting the matrix A above equal to the companion matrix C, we see that
Corollary 9.9.3. *Every polynomial of degree n with complex coefficients has a solution as a companion matrix*. ☐

This solution generates another solution given by the transpose C^T, which differs from C and is linearly independent of C always if the degree n > 2. ☐

9.10. Bring-Jerrard nilpotent additive hyperintricate polynomials.

A polynomial in X with real coefficients has the formal properties of a polynomial in x, where x is real or complex, see [Ro90]. In particular commutative properties are satisfied by this polynomial. A distinction is that X may be nilpotent, whereas x cannot be. By the definition of nilpotent

$$X^n = 0,$$

so that by taking determinants and using the multiplicative property of det

$$\det(X^n) = 0,$$

giving

$$(\det X)^n = \det X = 0.$$

In the non-nilpotent case we do not know for complex coefficients whether there is no solution by radicals for degree n > m to the Bring-Jerrard form of a polynomial

$$X^n + aX + b = 0,$$

whereas for a Bring-Jerrard or a more general polynomial in the nilpotent case

$$\det(X^n + aX) = \det(X)\det(X^{n-1} + a) = -\det(b),$$

which is not possible at all for complex b in characteristic zero (non-finite arithmetic), where we will have det X = 0, but the determinant of a non-zero complex number is non-zero. ☐

9.11. Symmetric matrices give constraints on polynomial coefficients.

We will show in chapter XI that if a matrix X is symmetric, then a similarity transformation PXP^{-1} can bring it to diagonalisable form. In this section we observe that this reduces for a polynomial in X of degree n to solving a series of non-matrix equations of degree n, implying constraints on the coefficients.

Theorem 9.11.1. *If a matrix X is symmetric, then a similarity transformation* PXP^{-1} *can bring it to diagonalisable form*. ☐

Theorem 9.11.2. *Symmetric matrix solutions give constraints on polynomial coefficients*.

Proof. Consider the equation

$$\sum_{i=0}^{n} T_i X^i = T_0 + T_1 X + ... + T_n X^n = 0 \qquad (1)$$

and apply a diagonalising transformation to give $A = PXP^{-1}$. Then

$$\sum_{i=0}^{n} T_i PA^i P^{-1} = T_0 PP^{-1} + T_1 PAP^{-1} + ... + T_n PA^n P^{-1} = 0, \qquad (2)$$

and multiplying by the non-singular P

$$\sum_{i=0}^{n} T_i PA^i = T_0 P + T_1 PA + ... + T_n PA^n = 0. \qquad (3)$$

A could be represented for example by

$$A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix},$$

so that

$$A^i = \begin{bmatrix} x^i & 0 \\ 0 & y^i \end{bmatrix}.$$

Since the T_i are general arbitrary matrices, so are the T_iP. Looking at, say a quadratic by way of demonstration, let

$$T_0 P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ T_1 P = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \ T_2 P = 1.$$

Then this reduces to a series of quadratic equations, for example

$$x^2 + a'x + a = 0, \quad (4)$$

$$x^2 + c'x + c = 0, \quad (5)$$

where for each equation for consistency there has to be a common root, which implies constraints on the coefficients. Note we can eliminate the term of highest degree and solve the two equations, obtaining a linear solution and a quadratic constraint on the coefficients by a resubstitution in say equation (4). This observation applies similarly to higher degree polynomial equations. □

9.12. Additive J-abelian solo zero polynomial theory.

We repeat the definition of J-abelian given in chapter II, section 2.14:

Definition 9.12.1. *An* n-*hyperintricate number is* J-*abelian* *if* U, V, ... W *are intricate numbers for the layers of the* n-*hyperintricate number* $\Sigma U_{V...W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*), J *is not real and* $J^2 = 0$ *or* ±1.

A general hyperintricate *cannot* be represented by X_Y where X and Y are separately J-abelian and J′-abelian respectively (X_Y has 8 components whereas a general 2-hyperintricate has 16).

For each layer of X and S_i if these are both J-abelian for the same J, then

$$XS_i = S_iX.$$

Conversely, if X and S_i are not mutually J-abelian, then

$$XS_i = R_iX \neq S_iX. \quad (1)$$

In this latter case R_i is dependent on X, and for degree n polynomials, the result of chapter I, 1.9 is needed recursively up to (i – 1) times to obtain from multiplicative format 9.5.(2) a left- or right-additive polynomial with coefficients T_i as in 9.5.(1).

If a hyperintricate number X is J-abelian then X^n is J-abelian. Thus unless X is real, or pure J for even powers, X^n is not J′-abelian, J′ ≠ J, since then any multiplicative factor would contain a non-zero coefficient with J.

If we allocate a polynomial of the form 9.2.(1) where T, U, ... W are real coefficients, or are J-abelian, and we select an intricate value of x which is not a multifunction, then this polynomial may be represented, as indicated in chapter XII, by using a J-abelian x of the form

$$x = e^{y + JK} + Je^{z + JL},$$

where y, z, K and L are real and J is a *constant* intricate number with $J = bi + c\alpha + d\phi$ and $J^2 = 0$ or ±1.

If v and w are intricate, with v J_1-abelian where $J_1^2 = 0$ or ±1 and w is similarly J_2-abelian, then for a polynomial in x we can set x = v + w where x is J-abelian, and we can put $x = x_1 + x_2$, where x_1 and x_2 are both J-abelian.

If a general solution by radicals on expanding out v + w were available, then there would be a solution for x, and therefore for expanding out in terms of $x_1 + x_2$.

For J ≠ 0 the polynomial in x acts as a 1-hyperimaginary, 1-hyperactual or 1-hyperphantom variable which is commutative in J, and maps bijectively to i, α or ϕ respectively. For a 1-hyperimaginary (complex) variable, solvability criterions if any apply. For a 1-hyperactual or

1-hyperphantom variable, x is not in a field, there not necessarily being a multiplicative inverse, but it is still a member of a polynomial ring and again solvability criterions apply. When J = 0 the algebra is also J-abelian.

For dimension n, n-hyperintricate numbers possess a maximal commutative subring. Thus for 2-hyperintricate numbers we can form commutative vectors with basis 1_1, J_1, 1_K and J_K, where $J^2 = 0, \pm 1$ and $K^2 = 0, \pm 1$, comprising

$$x = 1_1 e^{y_1 + J_1 L_1} + J_1 e^{z_1 + J_1 M_1} + 1_1 e^{y_2 + 1_K L_2} + 1_K e^{z_2 + 1_K M_2} + 1_1 e^{y_3 + J_K L_3} + J_K e^{z_3 + J_K M_3},$$

which is closed under addition and vector multiplication.

If $X = A_B$ satisfies a polynomial equation with real coefficients, then by symmetry so does $(A_B)^{\sim} = B_A$. The operator ~ is covariant: $(A_B)^{\sim}(C_D)^{\sim} = (A_B C_D)^{\sim}$. □

9.13. The classical additive complex quadratic.

We will now explore the consequences of equating complex parts in the classical solution of the quadratic. Let X, R and S be complex numbers, where X = a + bi, R = r + ti, S = s + ui, and

$$X^2 + RX + S = 0. \qquad (1)$$

We will use the method of equating complex parts, so the real and imaginary parts are equated separately to zero, giving

$$a^2 - b^2 + ra - tb + s = 0, \qquad (2)$$
$$2ab + rb + ta + u = 0. \qquad (3)$$

The solution on putting all imaginary coefficients, t and u to zero, gives the classical solution

$$a = -r/2,$$
$$b = \pm ½\sqrt{(4s - r^2)},$$

which can be extended to solutions with r and s complex.

On denying ourselves this luxury, we obtain the quartic equation

$$4a^4 + 4ra^3 + [5r^2 + 4s + (2r - 1)t^2]a^2$$
$$+ r[4(r^2 + s) + t^2]a + [r^2(r^2 + s) + rtu - u^2] = 0,$$

solvable also by classical means. Once again, these solutions can be extended to r, s, t and u complex. □

9.14. The additive quadratic in an intricate variable.

Method 1a. Firstly, let us see what equating intricate parts means for the quadratic. We describe the case of a quadratic in intricate variables with real coefficients. This case subsumes that for a quadratic in a complex variable with real coefficients. Solutions for real coefficients may be extended to coefficients in an intricate algebra. The constraints on this are revealed in the discussion of *Method 1b*.

We will set

$$X^2 + RX + S = 0, \qquad (1)$$

with R and S *real* numbers, where

$$X = a1 + bi + c\alpha + d\phi,$$

so

$$X^2 = (a^2 - b^2 + c^2 + d^2)1 + 2abi + 2ac\alpha + 2ad\phi.$$

Thus the real part gives

$$(a^2 - b^2 + c^2 + d^2) + Ra + S = 0, \quad (2)$$

and each intricate part gives

$$a = -R/2, \quad (3)$$

provided one of $b \neq 0$, $c \neq 0$ or $d \neq 0$ and thus

$$b = \pm[\surd(-(R^2/4) + S + c^2 + d^2)].$$

Hence

$$X = -(R/2)1 \pm[\surd(-(R^2/4) + S + c^2 + d^2)]i + c\alpha + d\phi.\square \quad (4)$$

Method 1b. If we wished to use more traditional methods, by killing central terms, then putting $X = Y + A$ gives under the assumption that Y and A mutually commute

$$Y^2 + (2A + R)Y + A^2 + RA + S = 0.$$

If we put

$$A = -R/2,$$

then

$$Y^2 = (R^2/4) - S.$$

The assumption that Y and A commute for

$$Y = p1 + qi + r\alpha + s\phi$$

and

$$A = f1 + gi + h\alpha + k\phi,$$

is

$$0 = YA - AY = 2\{(rk - sh)i + (qk - gs)\alpha + (gr - qh)\phi\}.$$

Eliminating r and q, on equating intricate parts, the ϕ coefficient is

$$g(sh/k) - (gs/k)\, h = 0,$$

which is equivalent to the two constraints of just

$$r = sh/k,$$
$$q = gs/k,$$

that is, the non-real parts of Y and A are in the same ratio. This is no more than that Y and A are in the same J-abelian format.

To obtain the intricate square root, we append

$$\Omega_2 = 1^{1/2} = e^{\pi k(bi + c\alpha + d\phi)},$$

where

$$b^2 = 1 + c^2 + d^2.$$

Then on putting $\pm_b$ as resulting from the square root of b^2 and $\pm_Y$ as resulting from the square root of Y^2, we have

$$X = -(R/2)1 \pm_b \pm_Y\surd[(-(R^2/4) + S)(1 + c^2 + d^2)]i \pm_Y [\surd((R^2/4) - S)][c\alpha + d\phi].$$

So transforming variables

$$c \to \pm_Y -d/[\surd(-R^2/4) + S],$$
$$d \to \pm_Y c/[\surd(-R^2/4) + S]$$

gives an equation equivalent to (4). $\square$

Method 1c. The method we have used for the quadratic as a standard method of killing central terms is in J-abelian format, which is the most general for J-abelian hyperintricate variables with real coefficients, other solutions of J-abelian type being supersets of this form.

What we have done so far is assume Y and A mutually commute, and have extended the solution by means of intricate roots of unity.

We will now explore for the quadratic true non-commutative solutions, *ab initio*. The method will parallel in other respects the solution by standard methods.

Let

$$A = a1 + bi + c\alpha + d\phi,$$
$$B = f1 + gi + h\alpha + k\phi$$

and

$$X = A + B.$$

Generally $AB \neq BA$, and we will not assume otherwise. Then equation (1) becomes

$$A^2 + [AB + BA + RA] + B^2 + RB + S = 0.$$

Now AB is the sum of symmetric and antisymmetric terms:

$$\begin{aligned} AB = {} & a1(gi + h\alpha + k\phi) \\ & + f1(bi + c\alpha + d\phi) \\ & + af - bg + ch + kd \\ & + (ck - hd)i + (bk - gd)\alpha + (bh - cg)(-\phi), \end{aligned}$$

the first three lines being symmetric: AB = BA. Thus if we consider AB + BA it consists of these first three lines only, doubled.

We will now kill the central term [AB + BA + RA]. Assume as an example that R and S are real.

We have, on equating intricate parts

$$2[af - bg + ch + kd] = -Ra,$$
$$2[ag + fb] = -Rb,$$
$$2[ah + fc] = -Rc,$$
$$2[ak + fd] = -Rd,$$

so

$$[(R/2) + f]^2 = -g^2 + h^2 + k^2. \qquad (5)$$

Then

$$A^2 + B^2 + RB + S = 0$$

becomes, on equating intricate parts

$$a^2 - b^2 + c^2 + d^2 + f^2 - g^2 + h^2 + k^2 + Rf + S = 0$$

or

$$a^2 - b^2 + c^2 + d^2 + 2[(R/2) + f]^2 - R^2/4 + S = 0$$

with

$$ab + [(R/2) + f]g = 0,$$
$$ac + [(R/2) + f]h = 0,$$
$$ad + [(R/2) + f]k = 0.$$

This leads to

$$a^2 + [(R/2) + f]^4/a^2 + 2[(R/2) + f]^2 - R^2/4 + S = 0,$$

which is solvable for a in terms of a free parameter f, which we can limit arbitrarily, if we do not want to solve by other methods, to $[(R/2) + f] = a$, where $b = -g$, $c = -h$ and $d = -k$, with k satisfying the constraint (5). □

9.15. The additive cubic in a complex and an intricate variable.

Method 2a. The method of equating intricate parts is useful because it does precisely that: it splits the solution into intricate parts. Again, the coefficients employed in the solution may be generalised as those in an intricate algebra. Let

$$X^3 + RX + S = 0,$$

with

$$X = a1 + bi + c\alpha + d\phi,$$
$$R = g1 + hi$$

and

$$S = t1 + ui.$$

Then

$$X^3 = (a^2 - 3b^2 + 3c^2 + 3d^2)a1$$
$$+ (3a^2 - b^2 + c^2 + d^2)bi$$
$$+ (3a^2 - b^2 + c^2 + d^2)c\alpha$$
$$+ (3a^2 - b^2 + c^2 + d^2)d\phi.$$

On equating intricate parts

$$(a^2 - 3b^2 + 3c^2 + 3d^2)a + ga - hb + t = 0, \quad (1)$$
$$(3a^2 - b^2 + c^2 + d^2)b + gb + ha + u = 0, \quad (2)$$
$$(3a^2 - b^2 + c^2 + d^2)c + gc - hd = 0, \quad (3)$$
$$(3a^2 - b^2 + c^2 + d^2)d + gd + hc = 0, \quad (4)$$

then if h = 0, (2) implies v = 0, so (1) gives

$$8a^3 + 2ga - t = 0, \quad (5)$$

and if $h \neq 0$, (3) and (4) imply

$$(3a^2 - b^2 + c^2 + d^2 + g)(c^2 + d^2) = 0, \quad (6)$$

and assuming real values for c and d implies

$$8a^3 + 2ga + hb - t = 0, \quad (7)$$
$$gb + ha + u = 0, \quad (8)$$
$$gc - hd = 0, \quad (9)$$
$$gd + hc = 0, \quad (10)$$

with the contradiction from (9) and (10) of $(c^2 + d^2) = 0$. Thus we must abandon the assumption that both c and d are real. (7) and (8) now give

$$8a^3 + 2ga - (h/g)a - (u/g) - t = 0. \ \square \quad (11)$$

It is possible a solution of (11) for a, obtainable by standard methods itemised next, may not be real. However a cubic in real variables always has one real solution. In other cases we have the option of selecting complex values of a.

Put

$$q = [(t/4) - (u/8t)]$$

and

$$r = (1/8)[(w/t) - v],$$

then [Ro90] set

$$a = y + z,$$

so that

$$a^3 = y^3 + z^3 + 3ayz.$$

Therefore

$$y^3 + z^3 + (3yz + q)a + r = 0. \qquad (12)$$

We now impose a second constraint:

$$yz = -q/3,$$

so that in (12) the linear term in a vanishes. We have

$$y^3 + z^3 + r = 0$$

and

$$y^3z^3 = -q^3/27.$$

These two equations can be solved for y^3 and z^3. In detail

$$y^3 – q^3/(27y^3) + r = 0,$$

and hence

$$y^6 + ry^3 – q^3/27 = 0,$$

with

$$z^6 + rz^3 – q^3/27 = 0.$$

The quadratic formula gives

$$y^3 = ½[-r + \sqrt{}(r^2 + 4q^3/27)]$$
$$z^3 = ½[-r – \sqrt{}(r^2 + 4q^3/27)].$$

If ω is a cube root of unity, the six cube roots available are

$$y,\ \omega y,\ \omega^2 y,\ z,\ \omega z \text{ and } \omega^2 z,$$

which may be paired to give a product –q/3

$$-q/3 = yz = (\omega y)(\omega^2 z) = (\omega^2 y)(\omega z).$$

We conclude that the roots of the cubic are

$$y + z,\ \omega y + \omega^2 z \text{ and } \omega^2 y + \omega z,$$

where

$$y = [\ ½(-r + \sqrt{}(r^2 + 4q^3/27))]^{1/3}$$

and

$$z = [\ ½(-r – \sqrt{}(r^2 + 4q^3/27))]^{1/3}. \ \square$$

Method 2b. We now extend method 2a to the case where R and S are intricate numbers. X retains the names of its intricate coefficients, but now

$$R = g1 + hi + j\alpha + k\phi$$

and

$$S = t1 + ui + v\alpha + w\phi,$$

and again

$$X^3 + RX + S = 0,$$

so that on putting $J^2 = -b^2 + c^2 + d^2$ on equating intricate parts

$$(a^2 – 3J^2)a + ga – hb + jc + kd + t = 0, \qquad (13)$$
$$(3a^2 – J^2)b + gb + ha + ck – dj + u = 0, \qquad (14)$$
$$(3a^2 – J^2)c + gc + ja + bk – hd + v = 0, \qquad (15)$$
$$(3a^2 – J^2)d + gd + ka – bj + hc + w = 0, \qquad (16)$$

and we have four variables a, b, c and d in the four equations (13) to (16). For b, c and d ≠ 0, on multiplying (13) by 3bcd, and subtracting acd × (14), abd × (15) and abc × (16) we obtain three equations in a^2, which enables us to derive two equations in a:

$$[(-8J^2 + 2g)(bj – ch) – jkc + j^2d + hkb – h^2d + ju + hv]a$$
$$+ 3[-hj(b^2 + c^2) + jkbd + j^2bc – hkcd – h^2bc – jtb – htc] = 0, \qquad (17)$$

and

$$[(-8J^2 + 2g)(bk - dh) - k^2c + jkd - hjb - h^2c - ku + hw]a$$
$$+ 3[-hk(b^2 + d^2) + k^2bd + jkbc - hjcd + h^2bd + ktb - htd] = 0. \qquad (18)$$
On eliminating a, we have an equation in b^5, and using (17) for a in (14), (15) and (16), we obtain four equations without a in b^5. The existence of the power b^5 in these solutions indicates this situation would be unsolvable by radicals if Galois restrictions were to hold. A result of chapter XI, section 11, enables a solution of the quintic by matrix approximation methods.

Going outside the intricate format, we can obtain a coefficient $b = b1_{1...1}$ replaced by 1_m, where m is a companion matrix satisfying the quintic in b^5, by the method of section 9.9. The matrix m may be represented hyperintricately by adding trailing diagonal 1's to m. Thus there also exists a hyperintricate solution to this problem. □

9.16. A multiplicative J-abelian composite zero quartic.

Let
$$(x + bi + c\alpha + d\phi)(x - bi - c\alpha + d\phi)(x - bi + c\alpha - d\phi)(x + bi - c\alpha - d\phi) = 0, \qquad (1)$$
where the J-abelian matrix $x = X_Y$ commutes with i, α and ϕ. This means that the layer X is real, whereas we will allow Y to be intricate. In this situation i is equivalent to i_1, α to α_1 and ϕ to ϕ_1. Since if x commutes each expression in brackets cannot be zero, equation (1) is a composite zero polynomial.

We will want to equate (1) to the polynomial of complex type
$$x^4 + px^2 + qx + r = 0. \qquad (2)$$

Then (1) becomes
$$x^4 + 2[b^2 - c^2 - d^2) - bd]x^2 - 8bd^2ix$$
$$+ (b^2 - c^2 + d^2)^2 + 4(b^2 - c^2 + d^2)bd\alpha + 4d^2(b^2 - c^2) - 8bcd^2\phi = 0. \qquad (3)$$

Reasons similar to 9.15, on setting $d\alpha = v$, $b + c = t$, $b - c = u$ and eliminating tu and bv, result in a sextic in v solvable by the general algorithms for polynomials of arbitrary degree of chapter XI, section 11.

Once again, as for the variable b in 9.15, there exists a hyperintricate allocation of v which commutes with other variables and for which there exists a solution by the method of section 9.9. □