Innovation in mathematics

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Jim H. Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, innovations or developments. It is also an encouragement to the aspiring mathematician to get involved in mathematical research at an early stage, and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBook *The climate and energy emergencies* was published in 2014, this eBook was also concluded in 2014 and *Superexponential algebra* will be released in 2015.

Foreword

This eBook describes some of my own works that constitute innovations in mathematics, a description, helpful to the interested reader, of the creative process in mathematics, and finally a translation into English of an article by Eisenstein that I have found full of astounding insight.

From the desire to understand the mathematics of the twentieth century, the eBook arose from the study of the mathematics of the nineteenth century and before, and progressively a commentary and some modifications of it. To the best of my knowledge, these innovations were never implemented in the twentieth century.

Creative mathematics – a viewpoint ties the themes of the book together. It is a light-hearted look at ways of creative thinking and implementing them in practice.

Discussion on ladder numbers and zero algebras provides for the general reader innovations in real number theory: a proof of the inconsistency of the uncountable continuum hypothesis versus countable Cauchy sequences for reals. Implicitly, this inconsistency has been thought a possibility since the work of Gödel. In more detail, a set S is countable with respect to a set T if there exists at least one bijection between S and T, but there may be other maps between the same sets which are not bijections. Using this definition of relative countability to look at Cantor's diagonal argument, we also give an example which does not generate a diagonal counterexample. We demonstrate by an extension of the proof of the countability of $\mathbb{Q}$ that the set $\{\mathbb{N}, \mathbb{N}^2, \ldots \mathbb{N}^p, \ldots\}$ is countable and we further show that superexponential operations, a generalisation of exponentiation, applied to the set of natural numbers $\mathbb{N}$, always generate sets isomorphic to $\mathbb{N}$, thus deconstructing the existence of a distinct power set. We introduce ladder numbers, which are are not well-ordered, implying the failure of a standard density argument of $\mathbb{Q}$ within them. We use methods of transfinite induction acting on the density of $\mathbb{Q}$ in $\mathbb{R}$ to demonstrate the inconsistency of currently axiomatised analysis, and provide an alternative. Many uses of ladder numbers are sketched.

Polynomial equations I – duplicate roots adds a nice twist to Galois theory – the sextic with duplicate roots is solvable by radicals, even though the quintic is not solvable directly by using Galois techniques. We indicate ways Galois theory may be generalised rigorously.

Polynomial equations II – transcendental solutions gives the general algorithm for solving polynomial equations of any degree by successive convergent approximations.

Introduction to intricate and hyperintricate numbers I develops the well-known idea that $\sqrt{-1}$ can be represented by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since complex numbers have two components, but 2×2 matrices have four, we develop a four dimensional object – an intricate number – that is an extension of complex numbers. This is generalised to $2^n \times 2^n$ matrices – hyperintricate numbers. Proofs using the algebra are given also in *Introduction to intricate and hyperintricate numbers II*, which takes this development technically further.

Some simple proofs of general reciprocity, after proving the quadratic reciprocity theorem, introduces a minor extension of the Euler totient theorem. It then uses the work of Eisenstein to develop ideas further, including for totient cubic and quartic reciprocity. Implications for the exponential algebra $\mathbb{D}1$, mentioned throughout the book, which differs from the standard in that $(e^{i\theta})^i = e^{i\theta}$, are also indicated.

An elementary investigation of the prime p = 4k – 1 *asymmetry theorem for quadratic residues I* investigates by elementary techniques the theorem known to hold by sophisticated transcendental techniques that for prime p = 4k – 1 the number of quadratic residues in the interval [1, 2k – 1] is greater than in the interval [2k, 4k – 2]. *Part II* of the same work introduces the concept of *parabolas* to describe this phenomenon. These elementary method ideas are related to the known result on the absence of a tenth discriminant, and a proof by these means would constitute a proof of this theorem by elementary methods.

Lastly, we provide a translation from French into English of Eisenstein's *Applications of algebra to transcendental arithmetic*.

I have found that to develop my results I have increasingly been using the undergraduate text *Transform linear algebra* by Frank Uhlig, and both from there and elsewhere his insights. I would like to thank a number of people for their comments on parts of the book, which changed some aspects of it: Roger Fenn, Doly García, Tim Gibbs, Roger Goodwin, Ben Greenfield, Paul Hammond and Steven Miller, and to James Hirschfeld, who stressed the importance of completing a project; this has become a permanent intention in my writing. The decisions made on the contents are my own, as is the responsibility for any errors.

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The third Davenport

Since a critic of my work suggested that my time should be sent on doing mathematical exercises and I had liked the work of Richard Guy and John H Conway (not John B Conway, but he is OK), *The Book of Numbers*, I chose an unsolved problem from Richard K. Guy's *Unsolved Problems in Number Theory* to keep him quiet. I chose one that looked number theoretic, because I did not like the specific nature of many problems there. It is the asymmetry problem for quadratic residues. I hope I am correct in saying I have solved it. The reader should check. It has been solved already, by Herman Weyl, I found, but the problem states that this should be solved by elementary methods, that is, not using complex numbers.

There is a book by Manin that says problems that use complex numbers are solvable when not using them isn't. I disagree. Complex numbers are linearly representable, so primitive methods should always work.

Well, it was clearly going to take longer than a fortnight, or a month or – oh – three years. I *was* persistent, and I learnt a lot on how not to get an answer. I wrote to Richard Guy, on thinking I had a solution when I had not, and I was surprised when he wrote back. Perhaps it was because I suggested the solution be put in the next edition of Unsolved Problems in Number Theory. I did not realise at the time that Richard was born on 30 September 1916. Looking at Wikipedia tonight, his age is 102. He agreed with me. I then got back to him and said I had not solved it. He said the answer was in Davenport.

Harold Davenport (Richard will have known him) was born on 30 October 1907 and died on 9 June 1969. I looked at his collected works, and could not find anything relevant (that I liked to solve it – too much about distributions). Then suddenly I realised that his book *The Higher Arithmetic* was at its eighth edition. It contained material on Wiles' proof of Fermat's Last Theorem. Clearly there was a second Davenport, but I still could not find anything relevant. The problem was *hard*. In fact it is equivalent to proving the tenth discriminant problem, and you do not use elementary methods for that. Elementary methods are difficult.

I branched out and looked at the problem in great detail to find a solution. The patterns are parabolas in the second paper. They are beautiful. God tells you they are there, and then mathematics tells you that you can work it out for yourself. It is entirely logical, so God, thanks for the practical issue, but when it comes to theory I am OK.

Well it wasn't solved this way either. I thought of writing to Richard saying there were two Davenports, but the answer wasn't in either. Here are the two Davenports.

First Davenport, in Iowa

Second Davenport, in Florida

Well, I gave up. So I wrote it up saying I had not solved the problem. Clearly I had pushed it further than anyone else. Perhaps another mathematician would take it up under the partial delusion that mathematics is about solving problems. In fact, it is entirely reasonable to work that way. It is just that I am not really interested in solving problems. They are incidentals to proving concepts are valid.

So I had not discovered the answer in either of the Davenports. Then, six months later, I discovered that I had solved the problem already, I just had not noticed it. I realised that I am the third Davenport. I can relax and go to the theatre.

Supplementary notes, May 2015, April 2017 and March 2018

Changes to this eBook are now frozen from March 2018 as a record of my thinking up to 2014. These supplementary notes were mainly created in May 2015. From April 2017 the work has contained Part III of *An elementary investigation of the prime* $p = 4k - 1$ *asymmetry theorem for quadratic residues* and from March 2018 a proof of the asymmetry theorem, with corresponding renumbering of the final chapter and additions to the index.

Where references are made to the work *Superexponential algebra* before it was completed, these references have been updated, and elsewhere. No other modifications have been made.

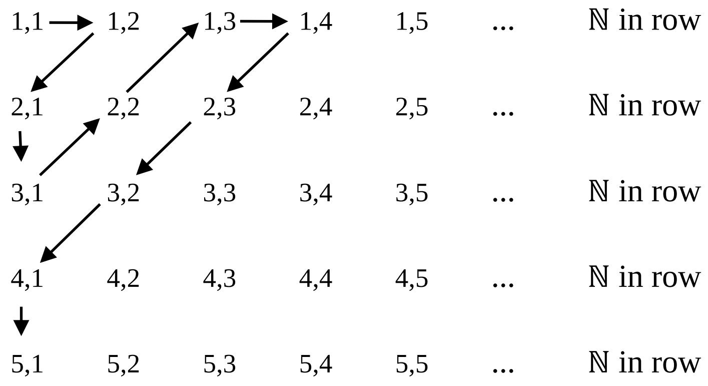
Intricate numbers are more usually given the name split-quaternions or coquaternions. They were introduced by James Cockle in 1849 under the latter name in the London-Edinburgh-Dublin Philosophical Magazine.

The argument used in *Discussion on ladder numbers and zero algebras* can be made more precise on page 02.19.

For finite or countably infinite sets S, T or U we define $\equiv$ by the existence of at least one bijection within the natural numbers $\mathbb{N}$. This is an equivalence relation, in that $S \equiv S$, if $S \equiv T$ then $T \equiv S$, and if $S \equiv T$ and $T \equiv U$, then $S \equiv U$, so this forms a partition between those sets belonging to the equivalence class, and those outside it. Then for sets S_n, T_n, $n \in \mathbb{N}$, if for each n $S_n \equiv T_n$, then for the set of all S_n, $\{S_{nn}\} \equiv \{T_{nn}\}$, where this means if $S_n \equiv T_n$, then the bijection is maintained for $S_{n+1} \equiv T_{n+1}$, and the second subscript indicates a distinguished copy for $S_n \neq \varnothing$, defined inductively: $S_{n1} = \{S_n\}$, $S_{n2} = \{\{S_n\}\}$, $S_{n\,p+1} = \{S_{np}\}$.

In this work we will adopt the argument of Cantor that the set $\mathbb{Q}$ of rationals is countable. Define the Cartesian product of all natural number pairs $\mathbb{N} \times \mathbb{N}$ as $\mathbb{N}^2$. Consider the rational numbers not in lowest terms given by the set $\mathbb{Q} \equiv \{\{1/n\}, \{2/n\}, \{3/n\}, \ldots\} \equiv$ the unordered distinguished copies$\{\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, \ldots\} \equiv \mathbb{N} \times \mathbb{N}$ (a set of ordered pairs) which by the Cantor argument given next is $\equiv \mathbb{N}$. The mapping from $\mathbb{N}$ to $\mathbb{Q}$ is given in the following diagram.

$(m, n) \to m/n \in \mathbb{Q}_{\neq 0}$.



What is meant by the symbols . . . in the sets just given? This indicates that if the pth position is occupied, then a similar item exists at position p + 1, although we can remove . . . from the language, and use the properties of $\mathbb{N}$ itself. Then by induction defined through the properties of $\mathbb{N}$, we have $\mathbb{N} \equiv \mathbb{N}^p$, for p a natural number, so that the set $\mathbb{N} \equiv \{\mathbb{N}_1, \mathbb{N}_2^2, \ldots, \mathbb{N}_p^p, \ldots\} \equiv$

$\{\mathbb{N}_1, \mathbb{N}_2 \times \mathbb{N}_2, \ldots, \mathbb{N}_p \times \mathbb{N}_p \times \ldots$ (p terms), $\ldots\}$ contains by definition $\mathbb{N}^{\mathbb{N}} \equiv \{\mathbb{N} \times \mathbb{N} \times \ldots\}$, in violation of the basic assumption of the continuum hypothesis in set theory, that $\{0, 1\}^{\mathbb{N}}$ is uncountable.

The description of the determinate zero algebra in the same paper is given on an axiomatic basis in the author's eBook *Superexponential algebra*.

The paper in session 11 and also the paper *Polynomial equations I – duplicate roots* assumes Galois *solvability* theory holds, which is investigated further in the eBook *Superexponential algebra*, where a description based on ring automorphisms is shown not to be equivalent to inner group automorphisms. We are not challenging Galois *representation* theory, a separate and distinct topic in multiplicative number theory. End results of Galois solvability theory are obtained from its replacement in the theory of varieties in the case of 'killing central terms', on eliminating dependencies.

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