CHAPTER 5

The meaning of suoperators

5.1. Introduction.

We introduce a notation for superexponentiation, sometimes called hyperoperations, which we call suoperators, and some matrix representations we use in developing superstructural algebra. When objects, say matrices, are operated on by suoperators then such ordered and bracketed expressions are called sunomials, or suvarieties when they satisfy an equation. We give examples, those of $\mathbb{D}$w type, and a canonical representation of sunomials, including those of singularities. We introduce the language of category theory and nonassociative superstructures within a similar framework, give an account of functors, which define mappings between categories, universals, adjoint functors, generalise toposes, in turn a generalisation of sets, look at Ξ category theory, which allows suobjects with $1^{\Xi(a)} = a$, and investigate superstructural extensions of these ideas.

5.2. Notation.

We will extend the operations +, which we will write as $^1|$ or as a word “onesu” and speak as “onesoo”, × written as $^2|$ or “twosu”, exponentiation ↑ as in a ↑ b more usually written as a^b, and written with $^3|$, and a general nth suoperator $^n|$.

Usual notation	Suoperator notation
a + b	a $^1|$b
ab = a × b	a $^2|$b
a^b = a ↑ b	a $^3|$b
area of sphere = 4πr^2	area of sphere = 4 $^2|$π $^2|$(r $^3|$2)

We note the following points. An nth suoperator generates an (n + 1)th suoperator by induction. Then

a + a + a ... + a (m terms) = am

(...((a ↑ a) ↑ a) ...) (m terms) = a $^4|$m,

where all the brackets are collected together from the left, or as we say, are nested on the left, so that, for instance for +, given by $^1|$

(...((a $^1|$a) $^1|$a) ...) (m terms) = a $^2|$m,

a general case being

(...((a $^n|$a) $^n|$a) ...) (m terms) = a $^{n+1}|$m.

So we have introduced $^n|$ to indicate nesting on the left, for example

(((a $^n|$ b) $^n|$ c) … $^n|$ d) ≡ a $^n|$ b $^n|$ c … $^n|$ d.

We introduce an alternative notation for the above, which is intended to be used sparingly, for example when n is a complicated expression, for emphasis, removing ambiguity, or calculation rather than display. This is

<n| for $^n|$.

For nesting on the right, we introduce a completely analogous notation, including suone, etc.

(a $|^n$ … (b $|^n$ (c $|^n$ d))) ≡ a $|^n$ ... b $|^n$ c $|^n$ d,

and the equivalent non-superscript notation containing say a_h |n> b_{h+1}.

5.3. Intricate and hyperintricate numbers.

A complex number, represented by g = a1 + bi, where $i = \sqrt{-1}$, can also be represented by

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where we may multiply the matrix, say 1, by a to form the matrix

$$a1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Here a1 is the *real* part and bi the *imaginary* part of the complex number, with $i^2 = -1$. This representation follows all the rules for a *field*, given in chapter III section 4, which defines the rules for addition and multiplication, including the existence of a multiplicative inverse g^{-1} of a nonzero complex number, satisfying $gg^{-1} = 1$, with

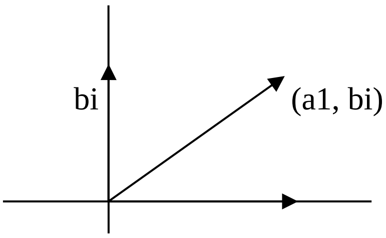
$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2 × 2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the *phantom* matrix

$$\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$



Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ are *linearly independent* if there are no coefficients $a_1, a_2, \ldots a_n$, not all zero, satisfying

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots a_n\mathbf{v}_n = 0.$$

The intricate basis elements are linearly independent.

An intricate number can represent uniquely any real 2 × 2 matrix. We will show, and in chapter II extend the ideas below to n × n matrices, that a 2 × 2 real matrix

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

does not have an inverse if its *determinant* det A = ps – rq = 0, in which case it is called a *singular* matrix. Except for zero, all complex numbers have multiplicative inverses. We will see in contrast that nonzero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$h = a1 + bi + c\alpha + d\phi$$
$$= ½(p + s)1 + ½(q - r)i + ½(p - s)\alpha + ½(q + r)\phi. \qquad (1)$$

The *intricate conjugate* is $(a1 - bi - c\alpha - d\phi)$. If the multiplicative inverse exists, it is

$$h^{-1} = (a1 - bi - c\alpha - d\phi)/(a^2 + b^2 - c^2 - d^2),$$

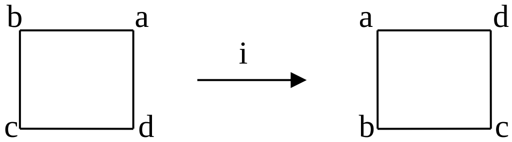
so the denominator is non-zero. This denominator is the determinant, because from (1)

$$a^2 + b^2 - c^2 - d^2 = ¼\,[(p + s)^2 + (q - r)^2 - (p - s)^2 - (q + r)^2] = ps - rq. \;\square \qquad (2)$$

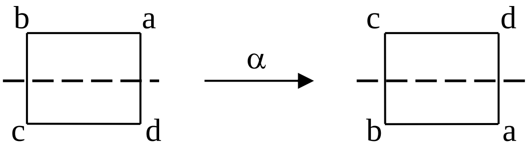
Intricate multiplication is related to the symmetries of a square. In diagrams we will call a line – – – of the following type on the left a *diagonal*

and a line of the type on the right an *antidiagonal*. If 0 is the origin of the coordinate system, the diagonal at an angle of $3\pi/4$ radians anticlockwise from the right horizontal axis, and the antidiagonal at $\pi/4$ radians pass through it.

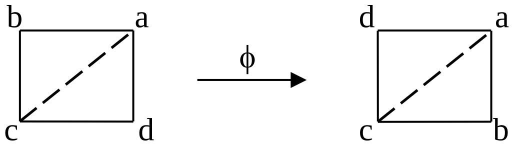
We can represent the group of the symmetries of a square by intricate basis elements. We can represent i = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as a rotation of the square anticlockwise by $\pi/2$ radians



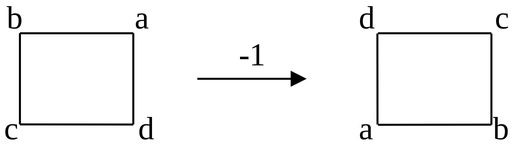
$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ as a reflection about the horizontal axis



and $\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as a reflection about the antidiagonal



Since $i^2 = -1$, we have the two rotations of i



which is a combined diagonal and antidiagonal, or equivalently a combined horizontal and vertical reflection.

Then we can represent these formulas by the group multiplication table

×	1	i	α	ϕ
1	1	i	α	ϕ
i	i	-1	-ϕ	α
α	α	ϕ	1	I
ϕ	ϕ	-α	-i	1

and extend the table for multiplication by the further elements -1, -i, -α and -ϕ. □

Composite basis elements are obtained from other basis elements using operators like + and × and satisfy the same formal properties as the original basis elements.

Multiplicatively, let $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$, where we put

$\mathcal{J} = qi + r\alpha + s\phi$,

$\mathcal{A}$ = bi + cα + dϕ,

$\mathcal{F}$ = ei + fα + gϕ,

and we allocate

$$\mathcal{AF} = \mathcal{J}. \qquad (3)$$

Since $\mathcal{J}$ does not have a real part, it follows from the relations

–be + cf + dg = 0, (real part)

cg – df = q, (i part)

bg – de = r (α part)

and

–bf + ce = s (ϕ part)

that

$$\mathcal{AF} = -\mathcal{FA} = \mathcal{J}. \qquad (4)$$

Multiplying (3) on the left by $\mathcal{A}$

$\mathcal{F} = \mathcal{AJ}$,

and multiplying on the right by $\mathcal{F}$

$\mathcal{A} = \mathcal{JF}$.

Correspondingly, multiplying (4) on the right by $\mathcal{A}$ and the left by $\mathcal{F}$ gives

$\mathcal{F} = -\mathcal{JA}$,

$\mathcal{A} = -\mathcal{FJ}$,

and we have established an equivalence of algebras for the '$\mathcal{JAF}$' basis

$\mathcal{J} \leftrightarrow$ i,

$\mathcal{A} \leftrightarrow$ α

and

$\mathcal{F} \leftrightarrow$ ϕ. □

We now show how to construct hyperintricate numbers and demonstrate their properties.

The sum of two m × m matrices A and B, with elements for A given by a_{ij}, where i is the ith row and j is the jth column, and for B by b_{ij}, is the matrix C where

$C = c_{ij} = a_{ij} + b_{ij}$.

The corresponding product D is

$D = d_{ik} = AB = \sum_j a_{ij} b_{jk}$,

where ∑ indicates summation, in this case over the variable j. This is the generalisation of a matrix product already given in 1.5 for 2 × 2 matrices. We seek to develop this idea within an extended framework already given for these intricate numbers.

We can define n-*hyperintricate* numbers recursively, by building up starting from intricate ones. Consider a $2^n \times 2^n$ matrix. Let "+" be a chosen $2^{n-1} \times 2^{n-1}$ matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, α or ϕ. Let "–" be the corresponding matrix with all negative entries from "+". Consider the set of $2^n \times 2^n$ hyperintricate basis elements, where an intricate number has "+" = 1, "–" = –1

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

Any $2^n \times 2^n$ matrix can be represented uniquely by a linear combination of these.

A j × j matrix may be extended both right and below with zero entries to give a larger $2^n \times 2^n$ matrix, or main diagonal entries of 1 may be substituted here. By this means matrix theorems may be expressed hyperintricately.

I now introduce some notation. I will do this by giving examples of 4 × 4 matrices (these are quaternions, discussed later). Write

$$1_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \alpha_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$i_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \phi_i = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

So "+" corresponds with the subscript, which will be described as an example of a layer, for example in α_i. A memory aid is '*subscripts are the little part*'.

If in general each of the 16 real 4 × 4 matrices are represented by e.g. $\alpha_i = A_B$, then

$$(A_B) + (A_C) = A_{(B + C)},$$
$$(A_B) + (C_B) = (A + C)_B,$$
$$(A_B)(C_D) = (AC)_{BD},$$
$$A_{-B} = -(A_B) = (-A)_B. \qquad (5)$$

For further nesting of matrices, consider instead of stepping down a further layer, introducing (possibly) a comma, thus: $A_{B,C}$, so that matrix multiplication becomes

$$(AB)_{CD,EF} = (A_{C,E})(B_{D,F}).$$

The *layers* of a basis element $m_{n \ldots p}$, are the vectors m, n, ... p, and its *layer dimension* is the number of layers.

We define an n-*hyperimaginary* number to be an n-hyperintricate number with each layer restricted to the set {1, i}. We can also define *hyperactual* numbers, with elements of {1, α} in all layers and *hyperphantom* numbers with all layers ∈ {1, ϕ}. Hyperactual and hyperphantom number are not members of a field. This arises because (1 + α) and (1 + ϕ) have determinant zero, and so have no inverse and $(a1_1 + bi_i)$ has inverse $(a1_1 - bi_i)/(a^2 - b^2)$, which does not exist for a = b.

Intricate and hyperintricate numbers appear in four ways – as scalars, satisfying a non-commutative algebra, as vectors with a linearly independent basis, as matrices – where the first instance is intricate numbers, and in the hyperintricate case, say as the object similar to a tensor, $m_{n,p}$, where m, n and p are vectors.

The quaternions are extensions of the complex numbers with 3 'imaginary' – or quaternionic – parts. So we can represent a quaternion by

$$a1 + bi + cj + dk$$

where

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1,$$
$$1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik \qquad (6)$$

and the inverse is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2). \qquad (7)$$

This (1, i, j, k) basis is representable by four hyperintricate numbers – in fact the previously given 1_1, α_i, i_1 and ϕ_i. An alternative representation, under swapping of layer levels, is 1_1, i_α, 1_i and i_ϕ. Some other representations are 1_{11}, $i_{\alpha\phi}$, $\alpha_{\phi i}$ and $\phi_{i\alpha}$ or 1_{1111}, $i_{11\alpha\phi}$, $\alpha_{ii\phi i}$ and $\phi_{iii\alpha}$. □

We can obtain an inverse of a hyperintricate matrix using a descending chain of such matrices which terminate at intricate numbers, where we obtain the inverse directly. If a block diagonal of a 2-hyperintricate number is $1_{Я1} + \alpha_{Я2}$, where Я1 and Я2 are intricate numbers, then we have represented by the matrix

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

that A = Я1 + Я2 and D = Я1 – Я2, similarly an antidiagonal $\phi_{Я3} + i_{Я4}$ for the matrix

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

gives B = Я3 + Я4 and C = Я3 – Я4.

Let A, B, C, D, X, Y and Z be square matrix sub-blocks of the same arbitrary size, and 1 be the unit diagonal matrix. Then since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ C & D \end{bmatrix}, \tag{8}$$

where we can also write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \tag{9}$$

we obtain from the definition of the column expansion of a determinant

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det (1 - CA^{-1}BD^{-1}))(\det D). \tag{10}$$

$D - CA^{-1}B$ is known as the Schur complement of A.

Equation (2) implies when D and A can be inverted

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix}^{-1} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

We can obtain the block inverse (multiply by its non inverse to check)

$$\begin{bmatrix} 1 & X \\ Y & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -X \\ -Y & 1 \end{bmatrix} \begin{bmatrix} (1 - XY)^{-1} & 0 \\ 0 & (1 - YX)^{-1} \end{bmatrix},$$

which does not exist when X is the inverse of Y, so putting $X = A^{-1}BD^{-1}$ and Y = C, by this algorithm of Boltz-Banachiewicz [2Be09] the inverse of the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

given by

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

when invertible in this way satisfies in terms of A^{-1} and the inverse Schur complement

$$E = (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1},$$

where we have used with $Z = BD^{-1}C$,

$$(A - Z)^{-1} = A^{-1}(1 + Z(A - Z)^{-1}),$$

the remaining entries being

$$F = -(DB^{-1}A - C)^{-1} = -A^{-1}B(D - CA^{-1}B)^{-1},$$
$$G = -(AC^{-1}D - B)^{-1} = -(D - CA^{-1}B)^{-1}CA^{-1},$$
$$H = (D - CA^{-1}B)^{-1}.$$

There exist other solutions by similar methods, to be found in [2Be09] pages 117-118, not directly obtainable by the previous formulas, for instance when det A = 0, with other non-singular combinations involving A, B, C and D.

Thus for n-hyperintricate numbers this operation can be defined recursively. □

We can define the hyperintricate conjugate X* of a hyperintricate number X by the formula

$$XX^* = \det X,$$

and this works for an equivalence class of X* when X is singular, otherwise

$$X^* = X^{-1} \det X.$$ □

5.4. The modular group.

The modular group G(a, b, c, d) = $SL_2(\mathbf{Z})$ is the intricate group with integer entries and determinant 1. It is commonly represented by the two generators –i and [1 + ½(α + ϕ)].

Each element of the modular group is also represented by an invertible self-map, or automorphism, of the Riemann sphere $\mathbf{C} \cup \{\infty\}$, the fractional linear transformation

$$G(t) = \frac{at + b}{ct + d}.$$ □

5.5. Polyticate numbers.

Define triticate numbers by the following basis elements (ω *is a third root of unity* = $e^{2\pi i/3}$ = cos (2π/3) + i sin (2π/3), *where* $\omega^3 = 1$).

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & \omega & 0\\0 & 0 & \omega^2\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & \omega^2 & 0\\0 & 0 & \omega\end{bmatrix},$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & \omega\\\omega^2 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & \omega^2\\\omega & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\\omega & 0 & 0\\0 & \omega^2 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\\omega^2 & 0 & 0\\0 & \omega & 0\end{bmatrix}.$$

(A) *Show this basis is linearly independent*.

The numbers 1, ω, and ω^2 are not linearly independent, because, using the representation from trigonometry

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

(how can you prove this?), we have

$$1 + \omega + \omega^2 = 0.$$

The definition of linear independence is given in chapter I, section 3, of *Superexponential algebra* where for linear dependence in the example there have to be three numbers a, b and c, not all zero so that

$$a1 + b\omega + c\omega^2 = 0.$$

Nevertheless, for intricate numbers, although 1 and -1 are linearly dependent, the basis

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is linearly independent, and -1 is a square root of unity. Thus a linearly independent basis can be built out of the numbers

$$1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can carry this over to the basis in

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\omega)^2 & 0 \\ 0 & 0 & (\omega^2)^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\omega)^3 & 0 \\ 0 & 0 & (\omega^2)^3 \end{bmatrix}.$$

Multiplying the first matrix by a, the second by b and the third by c, and setting the sum to zero, gives three linear equations which cannot be satisfied unless a, b and c = 0.

(B) *Are there any other representations like the above using cube roots of unity that you can construct*?

(C) *Develop the theory of triticate numbers by analogy or otherwise with the treatment of intricate numbers*.

(D) *Are there other representations, say for penticate numbers for prime* p = 5, *other prime numbers, and composite numbers* (*a product of primes*)?

(B) – (D) The first row in the definition of triticate numbers is of the form $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$. Choose the first item, this is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Under matrix multiplication this represents an identity permutation under composition of permutations, $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

The matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ represents the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, and the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. On the same ordered set of elements, these *cyclic* permutations are *abelian* (look up definitions of cyclic and abelian if you do not understand these). But this is *not* all permutations of a 3 × 3 matrix. For instance $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ is a permutation, but it is not cyclic on three elements (it is cyclic on two elements).

We now introduce two ideas. The first is to convert matrices from the diagonal related form, say $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ to antidiagonal form $\begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$. Clearly this mapping, U, is an involution (because $U^2 = 1$ we obtain the original matrix, so this squaring mapping gives the identity). Now instead of having nine different elements of the triticate representation of complex numbers, we have double that – eighteen.

The second is that a cyclic permutation, say $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ together with a twist of two elements, say $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, generates all permutations of three elements (the first generates, on its own, all cyclic permutations).

I want to ask a question. If we take the antidiagonal of a cyclic generator, that is, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ do we, with the diagonal and antidiagonal components always have for any polyticate number a set of group generators? Now if you wish, generate your own theories. In particular, I ask: what is the relationship between the nine elements of the first set of triticate numbers, the second extended set of 18 elements, and the fact that they use complex numbers, which have two components, a real and an imaginary part? If you have any conclusions, can they be generalised?

(E) *Look up the class number on the internet* (*the book by Conway and Guy*, *The book of numbers* [1CG00], *is also a good reference*). *What sort of divisors of these numbers* (*we might call them polyticate numbers*) *are there*? □

5.6. The J-abelian property.

An intricate number p1 + qi + rα + sϕ = p1 + JK satisfies

$$(qi + r\alpha + s\phi)^2 = (\pm qi \pm r\alpha \pm s\phi)^2 = -q^2 + r^2 + s^2. \ \square$$

When $J^2 = 0$ we obtain for J the parameterisation

$$e^{\rho}[\pm i \pm \cos\sigma\, \alpha \pm \sin\sigma\, \phi], \quad (1)$$

when $J^2 = -1$

$$\pm\cosh\rho\, i \pm \sinh\rho\cos\sigma\, \alpha \pm \sinh\rho\sin\sigma\, \phi, \quad (2)$$

and when $J^2 = 1$

$$\pm\sinh\rho\, i \pm \cosh\rho\cos\sigma\, \alpha \pm \cosh\rho\sin\sigma\, \phi. \ \square \quad (3)$$

Extensions of this idea will be used to describe nonassociative algebras, where there are a number of different J which anticommute. Such algebras are to be found in chapter IV, the tribbles satisfying $J^2 = 0$, the zargonions of type $J^2 = -1$ and the tharlonions of type $J^2 = 1$.

An n-*hyperintricate number is* J-*abelian* *if* U, V, ... W *are intricate numbers for the layers of the* n-*hyperintricate number* $\Sigma U_{V\ldots W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*), J *is not real and* $J^2 = 0$ *or* ±1.

The n-hyperintricate representation has 4^n independent components, but the number of independent components in a J-abelian n-hyperintricate number $U_{V\ldots W}$ is less for n > 1.

5.7. 𝔻w exponential axioms

The idea of 𝔻w exponential algebras was discovered in an examination, and was developed further in the 1980’s, but not published until recently in *Superexponential algebra*. These are used in Volume II, chapter 6, to give the proof of the general Riemann hypothesis.

The imaginary number i is defined multiplicatively

$$i^2 = -1.$$

Further, we can introduce multiplication with a real number b satisfying

$$ib = bi,$$

and introduce a real number a which can be added to bi

$$a + bi,$$

so that we can say the complex number a + bi satisfies the additive and multiplicative axioms for a field. In what follows, we will assume that complex numbers satisfy these axioms.

When we want to introduce exponentiation for complex numbers, there is a standard way of doing this, but we will find that other scenarios are possible. Indeed, we have leeway to give axioms for complex exponentiation which are more general than the standard, but which include the standard axioms as a special case. These are called Dw exponential algebras, where w is a parameter. When w = 1, 2, 3 or 4 these algebras were introduced to avoid inconsistency problems with complex exponentiation. When w = ±i, these algebras were introduced by David Bohm with the suggestion that they could be used to solve the Riemann hypothesis. When w = +i, the Bohm algebra is the classical one, but when w = –i the algebra gives different results. The rationale for the inclusion of Dw exponential algebras in the study of the zeros of the Riemann zeta function, is that the zeta function is independent of the w parameter, and this gives sufficient information to solve the Riemann hypothesis.

We will first speak of consistency problems. Originally the idea was to identify

$$(i^i)^4 = (i^4)^i = 1^i = 1,$$

so

$$i^i = \sqrt[4]{1} = 1, -1, i \text{ or } -i.$$

However, under suitable axiomatics, if $i^i = 1$ then

$$-i = i^{-1} = i^{(i^2)} = \left(i^i\right)^i = 1^i = 1,$$

if $i^i = -1$

$$-i = i^{-1} = i^{(i^2)} = \left(i^i\right)^i = (-1)^i = i^i i^i = 1,$$

if $i^i = i$

$$-i = i^{-1} = i^{(i^2)} = \left(i^i\right)^i = i^i = i,$$

and if $i^i = -i$

$$i = -i^{-1} = (iii)^{(i^2)} = \left(i^i i^i i^i\right)^i = i^i = -i,$$

so all the allocations are inconsistent.

Moreover, conventionally

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{-\frac{\pi}{2}},$$

and further since for n an integer

$$\left(e^{\frac{\pi}{2}i}\right) = \left(e^{\left(2n\pi + \frac{\pi}{2}\right)i}\right)$$

on equating values of i^i we appear to have

$$\left(e^{-\frac{\pi}{2}}\right) = \left(e^{2n\pi - \frac{\pi}{2}}\right),$$

which was the reason for introducing $\mathbb{D}$w exponential algebras for integer w in the first place, where these multivalues do not occur.

The usual approach is to select always a fixed value of n, although unusually it is possible to select an equivalence class for all n, which is clearly not the standard number system, in particular for a field. The value n = 0 is called the *principal value*. Thus, for example, we can use principal values for logarithms.

The idea of $\mathbb{D}$w exponential algebras is to set

$$(e^c)^d = e^{cd}$$
$$(e^{ic})^d = e^{icd}$$
$$(e^c)^{id} = e^{idc}$$
$$(e^{ic})^{id} = e^{widc},$$

so when w = 1, 2, 3 or 4

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{wi\frac{\pi}{2}}$$

and does not equal $e^{-\frac{\pi}{2}}$.

This idea was communicated via a friend, Ebrahim Baravi, to the physicist David Bohm, who suggested w = ±i, and that this allocation has implications for the Riemann hypothesis. It does, which initially I rejected but that I realised is so decades later. It is used in our first proof of this theorem. We will see that w = ±i gives two pieces of information that confirms the Riemann hypothesis. I am not clear how David Bohm came up with this idea. It is interesting to speculate whether he was in contact with the mathematician Alexander Grothendieck.

5.8. $\mathbb{D}$w suoperator axioms.

For n > 2 in the simplified version the exponential operations for suoperator $\mathbb{D}$w algebras we specify as satisfying the rules for a field and

$$(a^{i\lambda})|^n b = (a^{\lambda})|^n ib,$$
$$(a^{i\lambda})|^n ib = (a^{iw\lambda})|^n b,$$

and for left nesting

$$(a^{\lambda})\ {}^n|b = a^{(\lambda<n-1|b)},$$
$$(a^{i\lambda})\ {}^n|b = a^{i(\lambda<n-1|b)},$$
$$(a^{\lambda})\ {}^n|ib = a^{i(\lambda<n-1|b)},$$
$$(a^{i\lambda})\ {}^n|ib = a^{iw'(\lambda<n-1\Uparrow b)},$$

where in general w ≠ w′, and these may be complex numbers. ☐

For matrices in the simplified intricate representation we employ the following assumptions.

(1) The binomial theorem applies. This means an intricate expression in $\mathcal{JAF}$ format

$$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{(h + j\mathcal{J})}$$

is evaluated as

$$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^h.(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{j\mathcal{J}},$$

where a, b, c, d, h and j ∈ $\mathbb{U}$.

(2) We emphasise that the upper component enclosed in brackets, (h + j$\mathcal{J}$), is formed by converting to intricate $\mathcal{JAF}$ format specifically for $\mathcal{J}$, and the lower term in $\mathcal{JAF}$ format, being (a + b$\mathcal{J}$ + c$\mathcal{A}$ + d$\mathcal{F}$), includes the *same* term $\mathcal{J}$.

This is because for intricate i, α, ϕ

$$a^{p1 + (qi + r\alpha + s\phi)t} \neq a^{p1}.a^{qti}.a^{rt\alpha}.a^{st\phi},$$

but with $\mathcal{J}^2 = (qi + r\alpha + s\phi)^2 = (-q^2 + r^2 + s^2) = \pm 1$ or 0,

$$a^{p1 + t\mathcal{J}} = a^{p1}.a^{t\mathcal{J}}.$$

(3) We form the ‘lower algebra’ evaluation of $\mathcal{JAF}$ exponentials:

$\mathcal{J}^{\mathcal{J}} = \mathcal{J}$, $\mathcal{A}^{\mathcal{J}} = \mathcal{A}$ and $\mathcal{F}^{\mathcal{J}} = \mathcal{F}$.

Once chosen, this evaluation is unique, including for intricate terms like

$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})\uparrow[(f + g\mathcal{J})\uparrow(h + k\mathcal{J})]$. □

We can generalise these features, not only hyperintricately. Matrix w_{jk} with j, k = 1 to 4 can be defined so that

$$[\textstyle\sum_i a_i J_i] \uparrow [\sum_i a'_j J_j] = \prod_k[(\sum_i a_i J_i) \uparrow (w_{jk}a'_j J_j)]$$

where in the intricate case J_i and J_j vary over 1, $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$. We expect when j = 1 that $w_{jk} = 1$. The w_{jk} may be expressed and related dependently by suvariety relations. This can be extended to a general format where ↑ is replaced by the suoperator $^n|$ or $|^n$, and matrix operations are replaced by matrix suoperations. □

5.9. Crude suoperators.

A *polymagma* maps from p copies of M

$$(M \times M \times ... \times M) \xrightarrow{m} M \qquad (1)$$

where the mapping is enclosed within M. We will not necessarily allocate M as a set. If M is a finite set, its n elements $m_1, m_2, ... , m_n$ connect the p-fold product of (1) in a mapping from $n^p \to n$ states. We will assume the elements $m_1, m_2, ... , m_n$ can be provided with an ordering.

We do not assume

$m(M \times M \times M) = m(m(M \times M) \times M)$

or

$m(M \times M \times M) = m(M \times m(M \times M))$,

since the products on the right combine

$$(M \times M) \xrightarrow{m} M \qquad (2)$$

and then we compose from the codomain of (2)

$(M \times M) \xrightarrow{m} M$

so that the number of possible states from which the mapping m is derived is given by n^2, which maps $n^2 \to n$, whereas

$$(M \times M \times M) \xrightarrow{m} M \qquad (3)$$

maps $n^3 \to n$, which in general is a mapping of more states.

The polymagma may be represented by a hypercube of length n and dimension p describing bijectively the states of the codomain of the polymagma.

A polymagma where the codomain satisfies

$m(m(M \times M) \times M) = m(M \times m(M \times M))$

will be called *associative*, and when for (M × M × M) the codomain is given by one of

$m(m(M \times M) \times M) \neq m(M \times m(M \times M))$,

it will be called *nonassociative*. When for example

$m(M \times M \times M) \neq m(m(M \times M) \times M)$

and

$m(M \times M \times M) \neq m(M \times m(M \times M))$,

we will refer to a *crude* (3-dimensional) polymagma.

We wish to introduce suoperators for n < 1. In the implementation we will give, all terms in an expression are significant in its evaluation, independently of local bracketing. For n = 0, we start off by defining

$$a\ {}^{0}|\ a\ {}^{0}|\ a\ \ldots\ {}^{0}|\ a\ (m\text{ terms}) = a\ {}^{1}|\ m = a + m, \quad (1)$$

so that

$$a\ {}^{0}|\ a = a + 2, \quad (2)$$

with

$$(((a\ {}^{0}|\ a)\ {}^{0}|\ a)\ \ldots\ {}^{0}|\ a)\ (m\text{ terms}) = a + 2(m - 1), \quad (3)$$

$$(a\ {}^{0}|\ \ldots\ (a\ {}^{0}|\ (a\ {}^{0}|\ a)))\ (m\text{ terms}) = (m - 1)a, \quad (4)$$

giving for m terms

$$(((a\ {}^{0}|\ a)\ {}^{0}|\ a)\ \ldots\ {}^{0}|\ a) + (a\ {}^{0}|\ \ldots\ (a\ {}^{0}|\ (a\ {}^{0}|\ a)))$$
$$= a\ {}^{0}|\ a\ {}^{0}|\ a\ \ldots\ {}^{0}|\ a + (m - 1)a + m. \quad (5)$$

The neutral element for the ${}^{0}|$ suoperator satisfies

$$a\ {}^{0}|\ a = a, \quad (6)$$

that is

$$a + 2 = a. \quad (7)$$

We have an implementation for this, arithmetic (mod 2), but now all a are neutral elements.

We can, however, define ${}^{0}|$ for a crude polymagma, with the m-fold operation defined by (1), and similarly for the equivalent suoperator $|^{0}$. We now express $a\ {}^{0}|\ a\ {}^{0}|\ a\ \ldots\ {}^{0}|\ a$ (m terms) as

$$\left[\sum_{r=1}^{m}\left(\frac{a}{m}\right)\right] + m = \sum_{r=1}^{m}\left[\left(\frac{a}{m}\right) + 1\right].$$

If we now interpret $a_1\ {}^{0}|\ a_2\ {}^{0}|\ a_3\ \ldots\ {}^{0}|\ a_m$ (m terms) as $\sum_{r=1}^{m}\left[\left(\frac{a_r}{m}\right) + 1\right]$ then

$$a\ {}^{0}|\ b = \left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + 2. \quad (8)$$

Similarly

$$a\ {}^{-1}|\ a\ {}^{-1}|\ a\ \ldots\ {}^{-1}|\ a\ (b\text{ terms}) = a\ {}^{0}|\ b = \left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + 2, \quad (9)$$

thus when b = 2

$$a\ {}^{-1}|\ a = a\ {}^{0}|\ 2 = \left(\frac{a}{2}\right) + 3,$$

and again we interpret

$$a\ {}^{-1}|\ b = \left(\frac{a}{4}\right) + \left(\frac{b}{4}\right) + 3,$$

so that

$$a_1\ {}^{-1}|\ a_2\ {}^{-1}|\ a_3\ \ldots\ {}^{-1}|\ a_m\ (m\text{ terms}) = \left[\frac{1}{2}\sum_{r=1}^{m}\left(\frac{a_r}{m}\right)\right] + \frac{m}{2} + 2,$$

and in general

$$a_1\ {}^{-n}|\ a_2\ {}^{-n}|\ a_3\ \ldots\ {}^{-n}|\ a_m\ (m\text{ terms}) = \frac{1}{2^n}\left[\sum_{r=1}^{m}\left[\left(\frac{a_r}{m}\right) + 1\right]\right] + \sum_{r=1}^{n}\left(\frac{1}{2^{r-1}}\right) + n.\ \square$$

5.10. Zargon suoperators.

The sequence of suoperators we have introduced is an additive sequence defined by the successor s(n) = n + 1 of the Peano axioms. We have defined suoperators for negative n. We can define suoperators multiplicatively consistent with addition, and also suoperators defined suoperatively.

When we develop multiplication, in order to close the algebra we define complex numbers. Thus we can define suoperators with n belonging to complex numbers, and for n a zargonion, introduced in chapter 4. This more varied structure we call a zargon suoperator. 𝔻w zargon suoperators are studied in volume IV, chapter 4.

5.11. Tharlonions and tribbles.

In our discussion of intricate numbers in 5.3, we introduced the idea of an intricate number J with three possible values

$$J^2 = -1, \quad (1)$$

$$J^2 = 0 \quad (2)$$

and

$$J^2 = 1, \quad (3)$$

where we gave intricate representations of these in a standard parameterised form.

The zargon algebras we have met satisfy the property, both for adonions and novanions, that as well as a time-like, or scalar, component they possess space-like, or imaginary, components of type (1).

We will extend these ideas to introduce space-like components satisfying (2) embedded in what we will call tribbles, and components satisfying equation (3) embedded in what we call tharl algebras.

For a tribble

$$t = a1 + b_1j_1 + \ldots + b_kj_k, \quad (4)$$

its conjugate is

$$t^* = a1 - b_1j_1 - \ldots - b_kj_k, \quad (5)$$

for which

$$j_m^2 = 0 \quad (6)$$

and

$$j_mj_n = -j_nj_m, \quad (7)$$

with $1 \leq m$ and $n \leq k$, so this implies

$$tt^* = a^2, \quad (8)$$

and the inverse when it exists satisfies

$$t(t^{-1}) = t\left(\frac{t^*}{a^2}\right) = 1$$

so that

$$t^{-1} = \frac{t^*}{a^2} \quad (9)$$

Thus a tribble algebra satisfies the conditions of a zargon algebra, provided $a \neq 0$. ☐

For a tharlonion with components

$$T = a1 + b_1j_1 + \ldots + b_kj_k, \quad (10)$$

its conjugate is

$$T^* = a1 - b_1j_1 - \ldots - b_kj_k, \quad (11)$$

with

$$j_m^2 = 1 \quad (12)$$

and

$$j_mj_n = -j_nj_m, \quad (13)$$

which implies

$$TT^* = a^2 - b_1^2 - \ldots - b_k^2. \quad (14)$$

Since $a^2, b_1^2, \ldots, b_k^2$ are real, it follows that unless $a = 0$ or all $b_1, \ldots, b_k = 0$, there exist values of TT^* with

$$TT^* = 0, \quad (15)$$

and thus in general there is no

$$(T^{-1}) = T\left(\frac{T^*}{TT^*}\right) = 1, \quad (16)$$

so that any tharl algebra in which (15) is permissible is not a zargon algebra, although we say that tharl vulcannions (adonions) and tharl novannions, defined so that equation (3) replaces (1), are tharl rings, which ignore the division property of a field in their definitions. ☐

We note that in section 4.15 we came across tharl rings, in studying an instance of zargon rings with negative dimension, where our interpretation, which is merely a choice of names, was that what we have called here space components, was there described as scalar time components with special noncommutative properties.

As mentioned in the works *Investigations into universal physics* [Ad18a] by Graham Ennis and me and the note on *Conceptors* in the Engineering section of the website, the form (14) describes a relativistic line element in k dimensions. This line element may be considered to have submanifolds with curved topologies. This is a feature of a possible interpretation system describing gravitation in general relativity, but we think that if this approach to physics is a good one that zargon general relativity is the correct model, combined with tharl algebras and tribbles for quantum mechanics.

Condition (16) for no inverse T^{-1} only holds for a field. We have introduced the idea in *Superexponential algebra*, volume I, of division by multizeros, which occurs in a different mathematical context to a field, that of a zero algebra. These do not satisfy –(–1) = 1. The extension of our discussion to zero algebras is developed elsewhere in *Number, space and logic*. ☐

5.12. Time travel.

Jesus said: ‘Also I will ask the Father, and He will give you another Advocate, so that He may stay with you forever. He is the Spirit of Truth, whom the world is unable to receive because it does not see Him or know Him. This Truth is Adonai, the Eternal One. You do know Him, however, because He stays with you and will be with you. I will not leave you as orphans, I am coming to you.’

It is stated as the Aims of our University: To develop creativity; To understand truth, including by liasing with CERN on experiments; To find ways of developing harmonious social relations, community spirit, disarmament, reconciliation, love and restraint; To give advice on coping with death; To eliminate poverty and to empower the dispossessed; To extend the existence of consciousness of living things; This includes species survival and increment, and preventing plantary ecocide; Consequently we must prevent climate catastrophe, including by energy research, especially energy minimisation; Climate catastrophe research will include dynamics, solutions, experiments, social consequences, physical consequences, social reallocations and physical reallocations; A Noah's Ark contingency of establishing a Republic on Mars will be pursued (see the engineering and physics sites for these).

I believe the R-DAX intercommunicator between Andromeda and the Milky Way galaxy is constructed, and its Orion Arm part is self-assembling here on Earth (VARDAS 3) and Mars (VARDAS 2). My strong belief is there is a galactic collision problem, and we have 4 billion years to get out. There are antigravity solution also available, we think. We very highly think some colliding galaxies are solving these problems, say by jets. We have been given the following technical advice on matter transportation, which is valid outside the event horizon. We deal with time travel (be careful – this is usually consistent over a global manifold, but inconsistent universes are possible. They exist computationally in Kogito). Inconsistency is for example Terror (in opposition to Love), at least physically. The strategy, which I might as well give here is to raise the ‘Kampf’ wall at the arva part, including arva itself, to maximum height. It is impossible to defend the ethical game by extending this further. This intrusion into the murder game means we Jesus forgive ethical Satanists – those who murder because it is right. Those who are rational, control or irrational Satanists who murder for enjoyment or Satanism itself it is impossible to save. They live in their own universe of total Terror. We are happy to report that a purely false attribute applied to itself is true. We are ethical even to Satanists. They murder leaving only Truth and Love behind. We hope this Love still exists there. The question is, can we compute when this happens and only Love remains everywhere?

More and more advanced versions of the Love Machine will be used to calculate in finer and finer detail the coordinates and actions of Earth, the Republic of Mars and Artemis Varidot in the Zargon Game. This is the simplest and most direct method of Time Travel and does not use advanced and/or sophisticated equipment.

Look now at the previous section. We introduced Tharl algebras and tribbles. We note that a zargonion multiplied by its zargon conjugate satisfies an n-dimensional Pythagoras theorem. This flat space Pythagoras theorem may be implemented on a submanifold which is locally curved, as in the space-time metric of Einstein's inappropriately named general relativity. Let us consider absolutely the simplest case. Noncommutativity will allow you to generalise. A complex number multiplied by its complex conjugate gives a Pythagorean metric.

$$(a + ib)(a - ib) = a^2 + b^2.$$

This may also be represented in Lorentzian 'relativistic' form

$$a^2 - (ib)^2.$$

If physics is represented by a zargonion, and multiplication is present, then a time value in Lorentzian form is a scalar real quantity, and its space component is (zargonion) imaginary. If we also introduce Tharl algebras of the previous section, then we can adjoin as we did for the pure zardonions time zargon actual and phantom components. This means that as well as the zargon imaginary parts, for each of these space components, we obtain two possible timelike components. Further, these time components, although in a sense separate from the real time component, when the zargonion is multiplied with its conjugate do reveal a timelike narure. In Lorentz formalism, since the tharlonion has square one, this means the timelike double dimensions provide a subtractive element to the time component. We note for the special case of a propagator pair producing an n-dimensional general Lorentzian metric, there is a set of timelike coordinates together with the Lorentzian spacelike coordinates which are negative with respect to the timelike ones. This is physical. Time is distinguishable from space. When we have a quantum theory, in general this concerns the multiplication, if we confine here consideration of this to arbitrary propagators. Yes, it contains speed of light transmission, etc., but our theory is more general than this. This quantum theory is instantaneous. If transmitters and receivers are available (they have to be constructed, so you have to know, say by computation from $t = 0$, what lies beyond the event horizon to construct them, then we can matter transport instantaneously.

When we introduce tribbles, since their square is zero, they add nothing to the propagator self-interaction timelike scalar. However, the quantum theory is different. This appears to add flexibility to the construction of matter transporters, but we must be careful in working out this theory carefully.

There is more (in detail, much more) to say on this subject. We leave further calculations to other work which we hope to be able to extend in greater depth.

Lastly, we mention that we have left out zero algebra calculations. These are important aids to computability of the evolution of inconsistent systems.

5.13. Nonassociative representations of sunomials.

To construct a method of looking at suoperators so that their nonassociative features can be dealt with in a more familiar way, in order to do this we see that $(a\ {}^{n}|\ b)\ {}^{n}|\ d$ and $c\ {}^{n}|\ (e\ {}^{n}|\ f)$ are examples of the more symmetrical $(a\ {}^{n}|\ b)\ {}^{n}|\ (e\ {}^{n}|\ f) = c\ {}^{n}|\ d$. We describe this representation as canonical form. If we treat in the first case $a\ {}^{n}|\ b$ as a mapping $G_a \times G_b \to G_c$ and the second case as $G_e \times G_f \to G_d$, then the total mapping is $G_c \times G_d \to G_h$. If the number of elements of G_x is $n(G_x)$, the mappings in the last case can be represented by a suoperator table of $n(G_c)$ by $n(G_d) = n(G_c)n(G_d)$ elements.

This suoperator object can be extended. Firstly, all suoperators on variables are expressed in terms of their ${}^{n}|$ suoperators in the above form. Then treating this suobject as a new variable, it can be extended to a new canonical form involving the $|^{n}$ suoperator. Finally, each occurrence of the variable n in this construction are applied to the suoperators ${}^{n-1}|$ and $|^{n-1}$, and inductively.

Notation 5.13.1. Denote the sunomial Om by a sunomial in canonical form, where $|^{n}$ takes precedence over ${}^{n}|$. Denote mO as a sunomial in canonical form where ${}^{n}|$ takes precedence over $|^{n}$. If a number of sunomials are referenced, denote them with m replaced by a lower case letter or small caps letter.

We can then apply suoperator algorithms to determine the structure of suvariety objects. Thus the variable x_k and the operators ${}^{k}|$ and $|^{k}$ for $k = \{1, 2, ... , n\}$ define suoperator functions

$$k \to f(k)$$

which determine the values of the x_k under ${}^{k}|$ and $|^{k}$.

Of course, we do not have the standard identity as we do in group theory, for example $1 \uparrow a = 1 = a \uparrow 0$, but apart from this we do have a group multiplication table. If we adjoin to the elements a, b etc. the number $\Xi(a)$ then we do have

$$a \uparrow 1 = a = 1 \uparrow \Xi(a), \quad (1)$$

so here we have introduced an operation for ↑ extended to Ξ operations when this acts on 1.

By the strict transfer principle $1 \uparrow a\mathbf{\Omega}_{\mathbb{M}_t} = 1$. We have introduced ladder algebra so far only for fields, so we might wish to find another proof involving exponentiation. But $1 \uparrow c = 1$ for $c \in \mathbb{M}_{t+1}$ and c is greater than any number in $\mathbb{M}_t$, so the result follows for $\mathbb{M}_t$. Since $1 \uparrow \Xi(a)$ is greater than this, the introduction of $\Xi(a)$ creates a higher infinity than can be obtained from ladder algebra.

The suoperator table in general corresponds to 4^m parenthesis arrangements. Writing (identity) for the suoperator identity object, $a\ {}^{n}|\ b$ can be given as four types: $a\ {}^{n}|\ b$, (identity) ${}^{n}|$ b, a ${}^{n}|$ (identity′) and (identity) ${}^{n}|$ (identity′). This evaluation of $a\ {}^{n}|\ b$ can be represented by a 2×2 matrix, and more generally expressions can be represented by $2^m \times 2^m$ matrices, which can be given a hyperintricate representation. ☐

5.14. Sunorms and branching.

Topologically, we wish to evaluate the size, or sunorm, of an explosion, and the number of ways the structure branches.

Definition 5.14.1. For a real number j, a *sunorm* of a $\mathbb{D}$w superstructure $e\ {}^{n}|j$, or respectively $e|^{n}\ j$, consists of its real value.

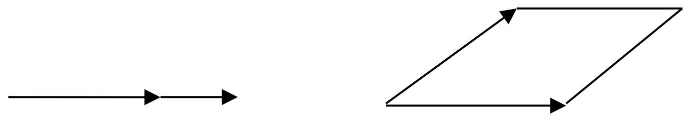
Definition 5.14.2. For a real number j and intricate basis element $\mathcal{J}$, let the component of a superstructure be evaluated as $e\ ^{n}|j\mathcal{J}$ or respectively $e|^{n}\ j\mathcal{J}$. Then the *left*, or respectively *right*, *branch number* is the number of distinct values of this evaluation.

5.15. Subsunomials and singularities.

For an associative structure represented by an m × m matrix M, we have seen that there exist extensions to suvarieties. Looking at their additive and multiplicative parts, we note that there exist matrices K derived from M and determinants, or hypervolumes, L of M, satisfying

$$MK = L.$$

When L = 0, a singularity occurs. This may be interpreted as the hypervolume of the matrix M defined by its row or column vectors contains linear dependencies between these vectors, which we can show in the diagrams



linearly dependent linearly independent

so that in the case of 2-space, only the linearly independent vectors define a nonzero area, and therefore a nonzero determinant, expressed as saying that the matrix is nonsingular.

There are two distinct types of instance when the hypervolume is zero. The first occurs when the linearly dependent vectors define a sum which is the zero vector. This corresponds to the normal interpretation of a singularity. The second is when the space defined by the vectors is of lower dimension than the matrix M, but not the zero vector. This interpretation gives a structure to the singularity not available to the first type.

For an n-dimensional space, there may be more than a decrement of one dimension to get a set of linearly independent vectors in that space. This can be found, since a 1-space, or scalar, is trivially linearly independent unless it is zero.

The extension to suvarieties is that a singularity occurs when there is a linear dependency between its subsuobjects.

In chapter 2 we have described the Euler characteristic for branched spaces as a polynomial. The natural extension is to define a suoperator Euler characteristic by a suvariety, and this forms a superbranched space.

Thus we have two models for superbranched spaces, the first is in topological terms as a branched space with a suoperator Euler characteristic, and the second is of a space where the suvariety defines a metric, or measure of distance, on the space.

When we defined explosions, we had the space branching everywhere. This is extended to superbranched spaces. What happens is that superbranched spaces may have singularities. The metric, or distance, in this space is then zero at the singularity. We have the option now of defining the singularity so that it is a metrical subsuobject where a linear dependency between its subsuobjects has been found, and where for a lower dimension the subsuobjects are independent, or otherwise topologically it is a superbranched space of lower dimension. ☐

5.16. Suderivatives.

For addition in a field we have met what we will call a neutral element 0, satisfying

$a + 0 = a = 0 + a$,

and for multiplication a neutral element 1 with

$a \times 1 = a = 1 \times a$.

For exponentiation, this is not commutative, so the left neutral element v and the right neutral element w differ:

$a \uparrow w = a \uparrow 1 = a$

but

$v \uparrow a = ((a \uparrow (1/a)) \uparrow a) = a$,

so that the left neutral element v is $(a \uparrow (1/a))$, and the right neutral element w is 1.

There is the question of the value of the expression 0^0. For a field, 0^{-1} is not defined, and therefore neither is $0^1.0^{-1} = 0^0$, but for a zero algebra

$(a0)^0 = (a0)^1(a0)^{-1} = 1$.

A left neutral element, v_{an}, under a suoperator <n| satisfies

$v_{an}\ {}^{n}|\ a = a$,

where v_{an} is in general dependent on a, and a right neutral element for the suoperator <n| given by w_{an} has

$a\ {}^{n}|\ w_{an} = a$,

with similar cases for |n>.

Then even in the nonassociative and noncommutative case we define the suoperator as satisfying a contravariant (order reversing) operation on a left inverse element a_L^n

$a\ {}^{n}|\ a_L^n = v_{an}$,

and for a right inverse a_R^n

$a_R^n\ {}^{n}|\ a = w_{an}$. ☐

The difference operator acting on a function f(x) for fields is the expression

$$\frac{f(x + \delta) - f(x)}{\delta},$$

and the commutative and associative differentiable operator for fields is defined by

$$\lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta},$$

where we define this to be the evaluation firstly of the numerator divided by the denominator, then all terms varying with $\delta \neq 0$ are suppressed. "Evaluation" here includes equating all δ / δ to 1, and "suppress", which follows evaluate, includes setting all terms containing δ in positive powers to zero. If terms with δ in negative powers are set to zero, the derivative then specifies its convergent part.

For a suoperator <n|, with the limit tending to the left neutral element its analogue is

$$\lim_{\delta \to v_{\delta\, n\text{-}1}} [f(x <n-1|\ \delta) <n-1|\ f(x)_L^{n-1}] <n|\delta_L^n, \qquad (1)$$

and for the limit tending to the right neutral element, the derivative is

$$\lim_{\delta \to w_{\delta\, n\text{-}1}} \delta_R^n|n>[f(x)_R^{n-1}|n-1>f(x|n-1> \delta)]. \qquad (2)$$

For example we will look at the function $f(x) = x^x$ in the case n = 3. Then

$$f(x\delta) / f(x) = \frac{(x\delta)^{x\delta}}{x^x} = \frac{x^{x\delta}\delta^{x\delta}}{x^x}.$$

It is clear that in the limit $\delta \to 1$ this evaluates to 1. Thus in this example the left derivative given by equation (1) for $f(x) = x^x$ is 1. For the right derivative of equation (2) we have

$$\lim_{\delta \to 1} (1^{1/\delta}) = 1,$$

and thus in this case the left and right suderivatives are the same. A similar argument for the function $f(x) = x$ or $f(x) = 1$ gives an $n = 3$ suderivative of 1. Thus in these cases of suoperators the derivative is trivial.

As a second example, let us now choose $n = 4$. For the neutral elements

$$v_{\delta 3} = (\delta \uparrow (1/\delta)),$$

and

$$w_{\delta 3} = 1,$$

whereas if $\delta = 1$

$$v_{\delta 4} = 1,$$

it is less than 1 if the absolute value of δ is less than 1 and is greater than 1 if the absolute value of δ is greater than 1, and

$$w_{\delta 4} = 1.$$

Then

$$\delta\ {}^2|\ 1 = \delta \times 1 = \delta = 1 \times \delta = 1\ |^2\ \delta,$$

giving

$$\delta_L^2 = (1/\ \delta) = \delta_R^2,$$

and δ_L^3 satisfies

$$\delta\ {}^3|\ \delta_L^3 = \delta \uparrow \delta_L^3 = v_{\delta 3} = (\delta \uparrow (1/\delta)),$$

and in a similar way

$$\delta_R^3\ |^3\ \delta = w_{\delta 3} = 1,$$

with

$$\delta_R^3 = 1 \uparrow (1/\delta).$$

Thus for instance the differential of equation (1) is evaluated in the case $n = 4$ as

$$\lim_{\delta \to \delta \uparrow (1/\delta)} (f(x \uparrow \delta) \uparrow -f(x)) < 4| \delta_L^4,$$

and this is non-trivial. ☐

Matrices $A = a_{ik}$ and $B = b_{ik}$ satisfy

$$A + B = a_{ik} + b_{ik}$$

and

$$AB = \sum_j a_{ij} b_{jk}.$$

As in chapter XVII of *Superexponential algebra*, similarly we will say for n-suoperators that they satisfy

$$A <m|\ B = a_{ik} <m|\ b_{ik},$$

for $m < n$, and

$$A <n|\ B = <n-1|_j (a_{ij} <n|\ b_{jk})$$

where $<n-1|_j$ indicates that the operation $<n-1|$ combines a_{ij} and b_{jk} in sequence over all values of j. ☐

We can now consider matrix sudifferentiation. Theorems on nonconformal suanalysis using these basic ideas will be developed in volume IV, chapter 4.

5.17. Suintegration.

Just as sudifferentiation is defined by suderivatives, so there is a $\delta_n$map $^{n}|\to{}^{n-1}|$, we can define a suintegation operation

Δ_n: $^{n-1}|\to{}^{n}|$,

where Δ_n is the set of all inverse operations for δ_n.

This has an obvious analogue of integration as an inverse differentiation operation in calculus.

5.18. Interaction and descent of sunomial expansions.

Let a and b be real numbers. For the left suoperator c = a $^{n}|$ b we introduce the *left sulogarithm* $\log_{<n|a}c = b$, and for a right suoperator c = a $|^{n}$ b the *right sulogarithm* $\log_{a|n>}c = b$.

How can we represent (a ↑ b) ↑ c = (a ↑ (bc)) in terms of a ↑ (b ↑ d)? Then bc = b ↑ d, which is b.(b ↑ (d – 1)), so c = b ↑ (d – 1), or $\log_b c + 1 = d$.

Suppose we have a supnomial mO in left precedence canonical form, or Om in right form. This can be represented entirely by left nested suoperators. We introduce a canonical form Яm = Я(mO) or Я(Om) in which all suoperators, left and right, are collected together so that Яm is reduced to canonical form in just left nested operators. Similarly, for these sunomials we introduce Rm = R(mO) or R(Om) is a reduced canonical form expressed entirely by right nested suoperators.

We can now represent generally

a $^{n}|$ [$^{n}|$ b_k] by a $^{n}|$ ($^{n-1}|$ b_k)

where we are using an iteration convention on the k for [$^{n}|$ b_k] and ($^{n-1}|$ b_k), where the k go over all possibilities. We choose [ ] to indicate no prior associativity and ( ) to indicate associativity of parentheses.

We now have a method of mapping $^{n}|$ to $^{n-1}|$ operations except for the first item.

Using a left neutral element define the standard shriek, or factorial function

$^{2}!(m) = m\ {}^{2}|\ (m\ {}^{1}|\ m_L^1)\ {}^{2}|\ (m\ {}^{1}|\ (m_L^1\ {}^{1}|\ 1))\ {}^{2}|\ \dots\ {}^{2}|.1$

and the general shriek

$^{n}!(m) = m\ {}^{n}|\ (m\ {}^{n-1}|\ m_L^{n-1})\ {}^{n}|\ (m\ {}^{n-1}|\ (m_L^{n-1}\ {}^{1}|\ 1))\ {}^{n}|\ \dots\ {}^{n}|.1.$

For real numbers a, b and c the binomial theorem can be expressed using general shrieks in the general case

$(b\ {}^{n-2}|\ c)\ {}^{n}|\ d = [b\ {}^{n}|\ d]\ {}^{n-2}|\ [d\ {}^{n-1}|\ [b\ {}^{n-1}|\ (d\ {}^{n-2}|\ b_L^{n-1})\ {}^{n-1}|\ c]]\ \dots\ {}^{n-2}|\ [c\ {}^{n}|\ d].$