CHAPTER 10

Consistent problems are solvable

10.1. Introduction.

In this chapter the symbol sea becomes rather choppy and threatens to engulf the boat. These details are directed at a higher aim: to prove that all consistent problems are solvable. With a view to subsequent generalisation, we extend our already found solution of the quintic to prove by comparison methods that the sextic is solvable. As a helpful adjunct to Jerrard theory which says the first three non-leading terms of a polynomial downwards from the second item in maximum degree to the next lowest degree can be eliminated, we show that there is a maximally compact form for the septic, having in equivalent full generality just an x^7 and x term in x in the polynomial. To end this stage of development, we provide two methods of solution for the septic equation, one by using polynomial wheel methods and one by comparison. Our task is then to generalise these ideas sufficiently to provide a proper induction proof within logical syntax reasoning to prove the general case that all finite polynomial equations have solutions in radicals.

Some of the consequences of this general principle are also given in this chapter, both specific extensions and general consequences such as the discussions on P/NP problems, which can be described with reference to Birkby's theorem on algorithmic solutions derived from considering the case of solutions of polynomial equations in a wider setting.

These proofs would not have been attempted without the insight that the results could be obtained. Perhaps initially this was due to no other fact than that I could see no purported proofs properly working as claimed. I also detected that there was no peer review of Galois theory which validated the claim that we were dealing with a result with good credentials. Perhaps I smelt a rat. Subsequent attempts to come to an opposite conclusion at first met with failure. The only reason I have made such an intense study of this problem was an inner compulsion to understand it. I did not require marks in an examination, and so there was no interior impulse to explain something of which I thought there was no explanation.

It is better than produce such detailed mechanisms as we develop here to look at insight as a formal mathematical property, to see whether we can establish a better method than syntax manipulation in the logic of reason. We must at this stage develop the syntax proof to ensure that insight is not its binary opposite, delusion. Nevertheless, insight back-propagates to syntax proofs. Viewed as insight, no syntax proof can be generated without the insight to specify its axioms, so that logical deduction must reconstitute to known theorems. An objective is therefore to generate an insight algebra. The existence of meanings on which insight depends can be formulated in a view of the world specified by general games.

10.2. The solution of the sextic given the solution of the quintic.

Our objective here is to document comparison methods that can be used, assuming the quintic is solvable, to solve the polynomial equation of degree six, the sextic. We find a solution dependent on a decic polynomial equation, which has degree ten, reducing to a quintic, thus solving the sextic when the quintic solution is known.

Consider the polynomial in form

$$x^6 + Gx^5 + Hx^4 + Kx^3 + Lx^2 + Mx + N = 0. \quad (1)$$

We will append by multiplication to this sextic polynomial the polynomial equation

$$x^4 + m_1x^3 + m_2x^2 + m_3x + m_4 = 0, \quad (2)$$

so that when multiplied together equations (1) and (2) result in

$$x^{10} + (G + m_1)x^9 + (H + Gm_1 + m_2)x^8 + (K + Hm_1 + Gm_2 + m_3)x^7$$
$$+ (L + Km_1 + Hm_2 + Gm_3 + m_4)x^6 + (M + Lm_1 + Km_2 + Hm_3 + Gm_4)x^5$$
$$+ (N + Mm_1 + Lm_2 + Km_3 + Hm_4)x^4 + (Nm_1 + Mm_2 + Lm_3 + Km_4)x^3$$
$$+ (Nm_2 + Mm_3 + Lm_4)x^2 + (Nm_3 + Mm_4)x + Nm_4 = 0. \quad (3)$$

Proceeding in a more elementary way than the method for the quintic, we compare this polynomial with the decic (degree 10) polynomial equation

$$(x^2 + px + q)^5 + a(x^2 + px + q)^4 + b(x^2 + px + q)^3 + c(x^2 + px + q)^2$$
$$+ d(x^2 + px + q) + e = 0, \quad (4)$$

with seven free coefficients, which expanded out becomes

$$x^{10} + 5(px + q)x^8 + 10(px + q)^2x^6 + 10(px + q)^3x^4 + 5(px + q)^4x^2 + (px + q)^5$$
$$+ a(x^8 + 4(px + q)x^6 + 6(px + q)^2x^4 + 4(px + q)^3x^2 + (px + q)^4)$$
$$+ b(x^6 + 3(px + q)x^4 + 3(px + q)^2x^2 + (px + q)^3)$$
$$+ c(x^4 + 2(px + q)x^2 + (px + q)^2) + d(x^2 + px + q) + e = 0,$$
$$x^{10} + 5px^9 + (5q + 10p^2 + a)x^8 + (20pq + 10p^3 + 4ap)x^7$$
$$+ (10q^2 + 30p^2q + 5p^4 + a(4q + 6p^2) + b)x^6$$
$$+ (30pq^2 + 20p^3q + p^5 + a(12pq + 4p^3) +3bp)x^5$$
$$+ (10q^3 + 30p^2q^2 + 5p^4 + a(6q^2 + 12p^2q + p^4) + b(3q + 3p^2) + c)x^4$$
$$+ (20pq^3 + 10p^3q^2 + a(12pq^2 + 4p^3q) + b(6pq + p^3) + 2cp)x^3$$
$$+ (5q^4 + 10p^2q^3 + a(4q^3 + 6p^2q^2) + b(3q^2 + 3p^2q) + c(2q + p^2) + d)x^2$$
$$+ (5pq^4 + 4apq^3 + 3bpq^2 + 2cpq + dp)x$$
$$+ q^5 + aq^4 + bq^3 + cq^2 + dq + e = 0. \quad (5)$$

Hence, on comparing coefficients between (3) and (5) we find

$$G + m_1 = 5p \quad (6)$$
$$H + Gm_1 + m_2 = 5q + 10p^2 + a \quad (7)$$
$$K + Hm_1 + Gm_2 + m_3 = 20pq + 10p^3 + 4ap \quad (8)$$
$$L + Km_1 + Hm_2 + Gm_3 + m_4 = 10q^2 + 30p^2q + 5p^4 + a(4q + 6p^2) + b \quad (9)$$
$$M + Lm_1 + Km_2 + Hm_3 + Gm_4 = 30pq^2 + 20p^3q + p^5 + a(12pq + 4p^3) +3bp \quad (10)$$
$$N + Mm_1 + Lm_2 + Km_3 + Hm_4 = 10q^3 + 30p^2q^2 + 5p^4$$
$$+ a(6q^2 + 12p^2q + p^4) + b(3q + 3p^2) + c \quad (11)$$
$$Nm_1 + Mm_2 + Lm_3 + Km_4 = 20pq^3 + 10p^3q^2 + a(12pq^2 + 4p^3q)$$
$$+ b(6pq + p^3) + 2cp \quad (12)$$
$$Nm_2 + Mm_3 + Lm_4 = 5q^4 + 10p^2q^3 + a(4q^3 + 6p^2q^2)$$
$$+ b(3q^2 + 3p^2q) + c(2q + p^2) + d \quad (13)$$
$$Nm_3 + Mm_4 = 5pq^4 + 4apq^3 + 3bpq^2 + 2cpq + dp \quad (14)$$
$$Nm_4 = q^5 + aq^4 + bq^3 + cq^2 + dq + e. \quad (15)$$

Of the seven variables p, q, a, b, c, d and e, and the four bogus roots m_1, m_2, m_3 and m_4 making 11 variables in all, we have 10 equations satisfying them, which gives us the freedom to set in the case of the sextic the now suitable $p = 1$. This is all we need to find a solution of equation (4), and its comparable equation (3) and hence (1). Putting $p = 1$ in equations (6) to (15) gives

$$G + m_1 = 5 \quad (16)$$
$$H + Gm_1 + m_2 = 5q + 10 + a \quad (17)$$
$$K + Hm_1 + Gm_2 + m_3 = 20q + 10 + 4a \quad (18)$$

$$L + Km_1 + Hm_2 + Gm_3 + m_4 = 10q^2 + 30q + 5 + a(4q + 6) + b \quad (19)$$
$$M + Lm_1 + Km_2 + Hm_3 + Gm_4 = 30q^2 + 20q + 1 + a(12q + 4) + 3b \quad (20)$$
$$N + Mm_1 + Lm_2 + Km_3 + Hm_4 = 10q^3 + 30q^2 + 5 + a(6q^2 + 12q + 1) + b(3q + 3) + c \quad (21)$$
$$Nm_1 + Mm_2 + Lm_3 + Km_4 = 20q^3 + 10q^2 + a(12q^2 + 4q) + b(6q + 1) + 2c \quad (22)$$
$$Nm_2 + Mm_3 + Lm_4 = 5q^4 + 10q^3 + a(4q^3 + 6q^2) + b(3q^2 + 3q) + c(2q + 1) + d \quad (23)$$
$$Nm_3 + Mm_4 = 5q^4 + 4aq^3 + 3bq^2 + 2cq + d \quad (24)$$
$$Nm_4 = q^5 + aq^4 + bq^3 + cq^2 + dq + e. \quad (25)$$

Then from (16) to (19) we are able to determine m_1 to m_4 directly
$$m_1 = 5 – G \quad (26)$$
$$m_2 = 5q + 10 + a – (5 – G)G – H \quad (27)$$
$$m_3 = 20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K \quad (28)$$
$$m_4 = 10q^2 + 30q + 5 + a(4q + 6) + b – (20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K)G – (5q + 10 + a – (5 – G)G – H)H – (5 – G)K – L. \quad (29)$$

Using these values we can express (20) to (25) so that they are eliminated. For instance, for equation (20)
$$M + L[5 – G] + K[5q + 10 + a – (5 – G)G – H] + H[20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K] + G[10q^2 + 30q + 5 + a(4q + 6) + b – (20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K)G – (5q + 10 + a – (5 – G)G – H)H – (5 – G)K – L] = 30q^2 + 20q + 1 + a(12q + 4) + 3b,$$
$$(G – 3)b = –M – L[5 – G] – K[5q + 10 – (5 – G)G – H] – H[20q + 10 – (5q + 10 – (5 – G)G – H)G – (5 – G)H – K] – G[10q^2 + 30q + 5 – (20q + 10 – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K)G – (5q + 10 – (5 – G)G – H)H – (5 – G)K – L] + 30q^2 + 20q + 1 + a(–K – 5H – G(4q + 2) + 12q + 4). \quad (30)$$

From (21) and (22) we can eliminate c
$$2N + (2M – N)m_1 + (2L – M)m_2 + (2K – L)m_3 + (2H – K)m_4 = 50q^2 + 10 + a(20q + 2) + 5b,$$
$$5b = 2N + (2M – N)(5 – G) + (2L – M)(5q + 10 + a – (5 – G)G – H) + (2K – L)(20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K) + (2H – K)(10q^2 + 30q + 5 + a(4q + 6) + b – (20q + 10 + 4a – (5q + 10 + a – (5 – G)G – H)G – (5 – G)H – K)G – (5q + 10 + a – (5 – G)G – H)H – (5 – G)K – L) + (–50q^2 – 10 – a(20q + 2)),$$

$$(5 – 2H + K)b = 2N + (2M – N)(5 – G) + (2L – M)(5q + 10 – (5 – G)G – H) + (2K – L)(20q + 10 – (5q + 10 – (5 – G)G – H)G – (5 – G)H – K) + (2H – K)(10q^2 + 30q + 5 – (20q + 10 – (5q + 10 – (5 – G)G – H)G – (5 – G)H – K)G – (5q + 10 – (5 – G)G – H)H – (5 – G)K – L) –50q^2 – 10 + a((2L – M) + 3(2K – L) + (2H – K) (4q + 2) – (20q + 2)). \quad (31)$$

Then using (30)

$$(5 - 2H + K)[-M - L[5 - G] - K[5q + 10 - (5 - G)G - H]$$
$$- H[20q + 10- (5q + 10 - (5 - G)G - H)G - (5 - G)H - K]$$
$$- G[10q^2 + 30q + 5$$
$$- (20q + 10 - (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 - (5 - G)G - H)H - (5 - G)K - L]$$
$$+ 30q^2 + 20q + 1$$
$$+ a(-K - 5H - G(4q + 2) + 12q + 4)] =$$
$$(G - 3)[2N + (2M - N)(5 - G) + (2L - M)(5q + 10 - (5 - G)G - H)$$
$$+ (2K - L)(20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K)$$
$$+ (2H - K)(10q^2 + 30q + 5$$
$$- (20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 - (5 - G)G - H)H - (5 - G)K - L) -50q^2 - 10$$
$$+ a((2L - M) + 3(2K - L) + (2H - K) (4q + 2) - (20q + 2))]. \qquad (32)$$

From (23) and (24) we can eliminate d

$$Nm_2 + (M - N)m_3 + (L - M)m_4 = 10q^3 + a(6q^2) + b(3q) + c,$$

using (21) for c we get

$$-N - Mm_1 + (N - L)m_2 + (M - N - K)m_3 + (L - M - H)m_4 =$$
$$-30q^2 - 5 - a(12q + 1) - 3b,$$

giving

$$-N - M[5 - G] + (N - L)[5q + 10 + a - (5 - G)G - H]$$
$$+ (M - N - K)[20q + 10 + 4a$$
$$- (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K]$$
$$+ (L - M - H)[10q^2 + 30q + 5 + a(4q + 6) + b$$
$$- (20q + 10 + 4a - (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 + a - (5 - G)G - H)H - (5 - G)K - L]$$
$$= -30q^2 - 5 - a(12q + 1) - 3b \qquad (33)$$

and then not sparing the graphic details, on substituting (30) for b in (33)

$$(G - 3)\{ -N - M[5 - G] + (N - L)[5q + 10 + a - (5 - G)G - H]$$
$$+ (M - N - K)[20q + 10 + 4a$$
$$- (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K]$$
$$+ (L - M - H)[10q^2 + 30q + 5 + a(4q + 6)$$
$$- (20q + 10 + 4a - (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 + a - (5 - G)G - H)H - (5 - G)K - L]$$
$$+ 30q^2 + 5 + a(12q + 1)\} =$$
$$(- 3 - L + M + H)\{-M - L[5 - G] - K[5q + 10 - (5 - G)G - H]$$
$$- H[20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K]$$
$$- G[10q^2 + 30q + 5$$
$$- (20q + 10 - (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 - (5 - G)G - H)H - (5 - G)K - L]$$
$$+ 30q^2 + 20q + 1 + a(-K - 5H - G(4q + 2) + 12q + 4)\},$$

which can be collected together in a as

$$(G - 3)\{ -N - M[5 - G] + (N - L)[5q + 10 - (5 - G)G - H]$$
$$+ (M - N - K)[20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K]$$
$$+ (L - M - H)[10q^2 + 30q + 5$$
$$- (20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 - (5 - G)G - H)H - (5 - G)K - L]$$
$$+ 30q^2 + 5 + a[(12q + 1) + (N - L) + (M - N - K)(4 + G)$$
$$+ (L - M - H)(4q + 6 - 4G + G^2 - 1)]\} =$$

$$(-3 - L + M + H)\{-M - L[5 - G] - K[5q + 10 - (5 - G)G - H]$$
$$- H[20q + 10 - (5q + 10 - (5 - G)G - H)G - (5 - G)H - K]$$
$$- G[10q^2 + 30q + 5$$
$$- (20q + 10 - (5q + 10 + a - (5 - G)G - H)G - (5 - G)H - K)G$$
$$- (5q + 10 - (5 - G)G - H)H - (5 - G)K - L]$$
$$+ 30q^2 + 20q + 1$$
$$+ a(-K - 5H - G(4q + 2) + 12q + 4)\}. \quad (34)$$

Thus we have two equations under suitable substitutions, from (32)

$$(sq^2 + tq + u) = a(vq + w) \quad (35)$$

and from (34)

$$(s'q^2 + t'q + u') = a(v'q + w'), \quad (36)$$

where by inspection the a terms do not become identically zero. This means

$$(v'q + w')(sq^2 + tq + u) = (vq + w)(s'q^2 + t'q + u'), \quad (37)$$

which is nontrivial since (34) contains qN but (32) does not, this is solvable as a cubic in q, and this almost concludes the calculation. Having found q from (37), knowing p = 1, a from (32), b from (31), c from (21), with m_1 to m_4 from (26) to (29) respectively, d from (24) and e from (25) we have obtained all coefficients of this decic given by the quintic equation (4). ☐

10.3. Maximally compacted form for the septic.

Since we have solved the quintic, and thus by section 12 the sextic, we suspect Jerrard form may be extended by the zeroisation of T in section 9.4 which follows, or more generally by T in arbitrary algebraic terms of U, V and W. We can go further than this.

Theorem 10.3.1. The septic can be reduced to general form

$$y^7 + V'y + W' = 0. \quad (1)$$

Proof. We have seen in section 8.2 a sextic can be solved when in the form

$$x^6 + Gx^5 + Hx^4 + Ix^3 + Jx^2 + Kx + L = 0. \quad (2)$$

Thus by hypothesis

$$(x^6 + Gx^5 + Hx^4 + Ix^3 + Jx^2 + Kx + L)(x + A) = 0$$

can be put in form (1) with transformations setting x = y, V′ = T and W′ = U, so

$$G + A = 0$$
$$H + GA = 0$$
$$I + HA = 0$$
$$J + IA = 0$$
$$K + JA = 0$$
$$L + KA = T$$
$$LA = U.$$

These give

$$A = -G \quad (3)$$
$$H = G^2 \quad (4)$$
$$I = -G^3 \quad (5)$$
$$J = G^4 \quad (6)$$
$$K = -G^5 \quad (7)$$
$$L - T = -G^6 \quad (8)$$
$$-LG = U. \quad (9)$$

Hence given a set of independent variables G, H, I, J, K and A, we can make a selection given by (3) to (9) to choose an L satisfying

$L^7 + TL^6 + U^6 = 0.$ (10)

so that making the substitution y = 1/L enables us to use the representation (1) directly.

Conversely if (10) holds so that we can choose a T satisfying it, then there exists a selection (3) to (9) of the variable A with the variables in (2) so that the equation is solvable. ☐

The existence of theorem 8.3.1 means there are a wide number of methods for solving equation (1). In the next section we will be rather arbitrary and choose the solution

$$x^7 + \left(\frac{W}{2} + 5\right)x + W = 0. \quad (11)$$

This is the end result of the first successful attempt at finding a solution. If y = rx then (11) reduces to (1) when

$$r = \frac{W'}{V'}\left(\frac{1}{2} + \frac{5}{W}\right). \ ☐ \quad (12)$$

10.4. A polynomial wheel method for solving the septic in radicals.

Now consider the octic variety generated by the polynomials

$x^4 + Ex^3 + Fx^2 + Gx + H$

and

$x^4 + Kx^3 + Lx^2 + Mx + N.$

Then the same method generates the solution to the general septic equation. Consider the variety

$$(x^4 + Ex^3 + Fx^2 + Gx + H)^2 - (x^4 + Kx^3 + Lx^2 + Mx + N)^2 = 0. \quad (1)$$

$$2(E - K)x^7 + (E^2 - K^2 + 2F - 2L)x^6 + 2(EF - KL + G - M)x^5 + (F^2 - L^2 + 2(EG - KM) + 2H - 2N)x^4 + 2(FG - LM + EH - KN)x^3 + (G^2 - M^2 + 2FH - 2LN)x^2 + (2GH - 2MN)x + H^2 - N^2 = 0,$$

$$x^7 + \left(\frac{E+K}{2} + \left(\frac{F-L}{E-K}\right)\right)x^6 + \left(\frac{EF-KL}{E-K} + \frac{G-M}{E-K}\right)x^5 + \left(\frac{F^2-L^2}{2(E-K)} + \frac{EG-KM}{E-K} + \frac{H-N}{E-K}\right)x^4 + \left(\frac{FG-LM}{E-F} + \frac{EH-KN}{E-F}\right)x^3 + \left(\frac{G^2-M^2}{2(E-F)} + \frac{FH-LN}{E-K}\right)x^2 + \left(\frac{GH-MN}{E-F}\right)x + \frac{H^2-N^2}{2(E-K)} = 0. \quad (2)$$

Put

$$E - K = H - N = p. \quad (3)$$

Equation (2) becomes

$$x^7 + \left(\frac{p+2K}{2} + \left(\frac{F-L}{p}\right)\right)x^6 + \left(F + K\left(\frac{F-L}{p}\right) + \left(\frac{G-M}{p}\right)\right)x^5 + \left(\frac{F^2-L^2}{2p} + E + K\left(\frac{G-M}{p}\right) + 1\right)x^4 + \left(\frac{FG-LM}{p} + H + K\right)x^3 + \left(\frac{G^2-M^2}{2p} + F + N\left(\frac{F-K}{p}\right)\right)x^2 + \left(G + N\left(\frac{G-M}{p}\right)\right)x + \frac{(p+2N)}{2} = 0. \quad (4)$$

Let us see what happens when this septic equation is represented in Jerrard form, which is equivalent to a general septic

$$x^7 + Tx^3 + Ux^2 + Vx + W = 0 \quad (5)$$

with a relationship to be specified between T, U, V and W. In fact later we will choose T = 0, as we are allowed to by theorem 8.3.1. We will also set U to zero, but not directly. We need this as an extra coefficient in our calculations. Theorem 8.3.1 means the general equation we will be aiming for is

$$y^7 + V'y + W' = 0. \quad (6)$$

Equating coefficients using equations (4) and (5)

$$0 = \frac{p + 2K}{2} + \left(\frac{F - L}{p}\right), \quad (7)$$

$$0 = F + K\left(\frac{F - L}{p}\right) + \left(\frac{G - M}{p}\right), \quad (8)$$

$$0 = \left(\frac{F - L}{p}\right)\left[\frac{1}{2}\left(\frac{F - L}{p}\right)p - p\left(\frac{F - L}{p}\right) + F\right] + p + K + K\left(\frac{G - M}{p}\right) + 1, \quad (9)$$

$$T = F\left(\frac{G - M}{p}\right) + M\left(\frac{F - L}{p}\right) + p + N + K, \quad (10)$$

$$U = \left(\frac{G - M}{p}\right)\left[\frac{1}{2}\left(\frac{G - M}{p}\right)p + M\right] + F + N\left(\frac{F - K}{p}\right), \quad (11)$$

$$V = M + (p + N)\left(\frac{G - M}{p}\right), \quad (12)$$

$$W = \frac{(p + 2N)}{2}. \quad (13)$$

Thus

$$\left(\frac{F - L}{p}\right) = -\frac{p}{2} - K, \quad (14)$$

$$\left(\frac{G - M}{p}\right) = -F + K\left(\frac{p}{2} + K\right), \quad (15)$$

$$0 = \left(-\frac{p}{2} - K\right)\left[\frac{1}{2}\left(\frac{p}{2} + K\right)p + F\right] + p + K + K\left(-F + K\left(\frac{p}{2} + K\right)\right) + 1,$$

$$F\left(\frac{p}{2} + 2K\right) = -\frac{1}{2}\left(\frac{p}{2} + K\right)^2 p + K^2\left(\frac{p}{2} + K\right) + p + K + 1, \quad (16)$$

$$N = W - \frac{p}{2}, \quad (17)$$

$$M = V - \left(W + \frac{p}{2}\right)\left(-F + K\left(\frac{p}{2} + K\right)\right), \quad (18)$$

$$T = F\left(-F + K\left(\frac{p}{2} + K\right)\right) - \left[V - \left(W + \frac{p}{2}\right)\left(-F + K\left(\frac{p}{2} + K\right)\right)\right]\left(\frac{p}{2} + K\right)$$
$$+ p + W + \frac{p}{2} + K, \quad (19)$$

$$U = \left(-F + K\left(\frac{p}{2} + K\right)\right)$$
$$+ \left[\frac{1}{2}\left(-F + K\left(\frac{p}{2} + K\right)\right)p + V - \left(W + \frac{p}{2}\right)\left(-F + K\left(\frac{p}{2} + K\right)\right)\right]$$
$$+ F + \left(W - \frac{p}{2}\right)\left(\frac{F - K}{p}\right), \quad (19)$$

so two F's and two $\frac{1}{2}$Fp's cancel in (19).

Three operative variables remain in equations (16), (19) and (20). We will solve them in a similar way that we dealt with equations (15), (16) and (17) in the previous section. Consider the three variables F, K, and $Y = \frac{p}{2} + K$.

Equation (16) becomes

$$Y^3 = -F(Y + K) + KY^2 + (K^2 + 2)Y - K + 1, \quad (21)$$

equation (18) is

$$T = F(-F + KY) - [V - (W - K + Y)(-F + KY)]Y$$
$$+ W - 2K + \frac{3}{2}Y, \quad (22)$$

and

$$U = KY + \left[\frac{1}{2}(-F + KY)p + V - (W - K + Y)(-F + KY)\right]$$
$$+(W + K - Y)\left(\frac{F - K}{2Y - 2K}\right),$$
$$2U(Y - K) = -(3(Y - K) + W)F + (Y - K)(KY + 2V + 2KWY + K) - KW + V. \quad (23)$$

Multiply (21) by K and subtract (22)

$$(-WK + F)Y^2 + \left(-K^2 - 2K + WF - V - KF - \frac{3}{2}\right)Y$$
$$+ FK^2 + K^2 + F^2 - W + K + T. \quad (24)$$

The two equations (21) and (23) in terms of F are linear in this variable. We wish to obtain from (21) and (23) an equation with no F terms which is cubic in K. From these

$$-Y^3 + KY^2 + (K^2 + 2)Y - K + 1)(3Y - 3K - W) =$$
$$[(1 + 2W)KY^2 + (-2U + 2V + K - K^2 - 2K^2W)Y$$
$$+2UK - 2KV - K^2 - KW + V](Y + K). \quad (25)$$

From (25) we obtain by appending a linear factor from (21) and (23)

$$(K + \alpha)Y^3 + [(W - K)K - F - \alpha K]Y^2 +$$
$$[(W - K)(-F) + V + KF + \frac{3}{2} + \alpha F - \alpha(K^2 + 2)]Y +$$
$$- FK^2 - K^2 - F^2 - W + K + T - \alpha K + \alpha = 0. \quad (26)$$

10.5. Stepping up solvability of odd degree polynomial equations.

This result is part of a more general one, that when the degree n of a solvable polynomial is odd, so all polynomials of degree < n are solvable, then the polynomial of degree (n + 1) is solvable.

We saw that for n = 3 the comparison method can be used to create a solution by radicals of a polynomial equation of degree 2n from a solvable equation of degree 2n – 1. This is part of a more general theorem holding for an arbitrary natural number n. We then continue by appending to the first equation 2n – 2 spurious roots, so that the resulting degree is 4n – 2. The solution is dependent on a comparison polynomial equation of degree 4n – 2, which reduces to a polynomial in 2n – 1 variables, each variable of which is a quadratic, so that in total for this polynomial the number of independent variables is 2n + 1. The total number of comparison variables minus the number of spurious roots is (2n + 1) – (2n – 2) = 3, which on the face of it looks very manageable. The main difficulty arises in designing a notation in which an induction can be followed for a large number of variables. ☐

10.6. Compacted form for general polynomials.

10.7. The solution of a general polynomial equation in radicals.

Further, any equation of lower degree than this can be solved by adjoining bogus roots.

This is the general polynomial wheel method solving polynomial equations of arbitrary large degree. ☐

10.8. Ladder solutions.

10.9. Solutions for Gaussian integer degrees.

Any root $(x^{a + bi} + c) = 0$ may be multiplied by $(x^{a - bi} + c)$ to give a root of a polynomial of real degree. Thus by appending such roots to a polynomial in Gaussian integer degree in either additive or multiplicative format, a polynomial in integer degree can be found, and a reversible algorithm can be applied to obtain a bijection in radicals between additive and multiplicative format polynomials in this case. This method can also be extended to Heegner number degrees, and indeed any general polynomial with degree surds in their real and complex parts. ☐

10.10. General elliptic curves and polynomial wheels.

We have seen in section 11 that the quintic is related to an elliptic curve, and we have now solved this problem. Thus there is a connection between elliptic curves, and their higher dimensional analogues, and polynomial wheel methods, so we can use polynomial wheels to give answers to problems formed in terms of higher dimensional elliptic curves. ☐