CHAPTER 1

Trees and amalgams

1.1. Introduction.

The most important part of mathematics is not solving problems, but correctly stating them.

Since the modularity theorem in the first instance and zeta functions deal with associative algebras, *insight* will tell us we can deal with this using undirected graphs. For the corresponding *zargon* case, insight will inform us to look at directed graphs. We are dealing with polynomials, but not in the suoperator case with normal graphs. This should eventually lead to glyphs.

1.2. Undirected graphs.

Definition 1.2.1. An *undirected graph* Δ consists of a set X of *vertices* and a set Y of *edges*, with two maps

$Y \to X \times X$: $\quad y \to (\text{origin}(y), \text{terminus}(y))$

$Y \to Y$: $\quad y \to \text{distance}(y)$

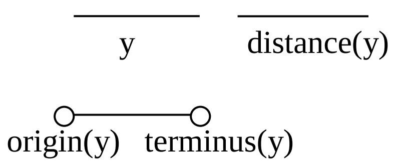
which for every $y \in Y$ satisfy the conditions

distance y = y

and

origin(y) ≠ terminus(distance(y)),

origin(y) = terminus(terminus(distance(y))).



1.3. Cubic graphs.

The polynomial wheel theory of Vol I, particularly starting from 7.13, shows for solvability we are interested in polynomials rather than groups, and a question now is, since any infinite group can be represented by an infinite graph, and correspondingly finitely, what is the relationship between polynomials and graphs?

We will look at the currently understood review of results for groups and graphs below. We then need to know whether the results are correct.

https://en.wikipedia.org/wiki/Cubic_graph

1.4. Frucht's theorem.

https://en.wikipedia.org/wiki/Frucht%27s_theorem

1.5. Directed graphs.

Definition 1.5.1. A *directed graph* Γ consists of a set X of *vertices* and a set Y of *edges*, with two maps

$Y \to X \times X$: $y \to (\text{origin}(y), \text{terminus}(y))$

$Y \to Y$: $y \to \text{reverse}(y)$

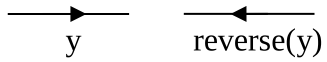
which for every $y \in Y$ satisfy the conditions

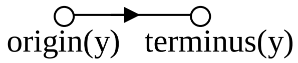
$\text{reverse } y \neq y$,

$\text{reverse}(\text{reverse}(y)) = y$

and

$\text{origin}(y) = \text{terminus}(\text{reverse}(y))$.





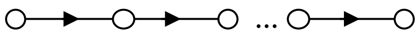
Any collection of objects and morphisms defines a (possibly large) directed graph *G*.

It seems that whereas we are happy to deal with the modularity theorem and its consequences for undirected graphs, for zargon algebras, we must concern ourselves with directed graphs.

1.6. Trees.

We first define trees [Se00].

Definition 1.6.1. A *path* is a finite sequence of edges $y_1, y_2, \ldots, y_n$ with $\text{terminus}(y_i) = \text{origin}(y_{i+1})$, $i < n$. The path may be said to have an origin, terminus pair (y_1, y_n).



Definition 1.6.2. A *circuit* is a path with $\text{terminus}(y_n) = \text{origin}(y_1)$.

A circuit remains a circuit under a cyclic permutation of the y_i, since the circuit may be defined from a new path with origin y_k for some k.

Definition 1.6.3. A directed graph is said to be *connected* if any pair of vertices is the origin, terminus pair of at least one path.

Definition 1.6.4. A *tree* is a connected nonempty directed graph without circuits.

Definition 1.6.5. A *node* is an origin or a terminus in a tree. A *parent node* or *parent* is an origin in a tree. A *child node* or *child* is a terminus in a tree. A *root* of a tree is a child with no parent, a *leaf* of a tree is a parent with no child.

1.7. Free groups. [Ar88]

Definition 1.7.1. A subset X of a group G is a *free set of generators* for G if every $g \in G$ not equal to the identity can be uniquely expressed as a finite product

$$g = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

where the x_i are distinct and $m_i \in \mathbb{N}_{\neq 0}$.

Definition 1.7.2. A *word* in the alphabet X is a finite associative product

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$
where an *empty word* is the word x^0 with no symbols.

Definition 1.7.3. A *reduced word* satisfies the property that the x_i are distinct and $m_i \in \mathbb{Z}_{\neq 0}$.

The sequence of words x_1x_2 may have a common factor between the last word of x_1 and the first word of x_2, but may be expressed as the reduced word $\overline{x_1 x_2}$.

Theorem 1.7.4. *Reduced words form a group*.

Proof. It is associative
$$\overline{\overline{(x_1x_2)}x_3} = \overline{x_1\overline{(x_2x_3)}} = \overline{x_1x_2x_3},$$
it has an identity, the empty word $\overline{x^0} = x^0$, and reduced words in the group have an inverse
$$\overline{\left(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\right)} \times \overline{\left(x_1^{-m_n} x_2^{-m_{n-1}} \cdots x_n^{-m_1}\right)} = x^0. \ \square$$

Corollary 1.7.5. *Reduced words are uniquely expressed*.

Proof. This follows from the fact that elements of a group are unique. ☐

Definition 1.7.6. The group of reduced words over the alphabet X is called the *free group generated by the elements of* X and is denoted by F(X).

Theorem 1.7.7. Let X be a subset of the group G.

1.8. The Nielsen-Schreier theorem. [Ar88]

1.9. Branched retracts.

1.10. Amalgams.