CHAPTER 3

Conformal and nonconformal analysis

3.1. Introduction.

The hyperintricate Cauchy-Riemann equations and the complex Cauchy integral formula are considered.

3.2. Fourier series. [SS03]

Fourier series are used to describe vibrating strings, travelling waves and heat flow. We note that the states of a system show what is there, but for practical applications transformations of states can describe what is observed. Our approach here is the logical one of dealing with the states of the system first, even if this leads to delayed gratification in describing practical examples as transformations.

The definition of the exponential function

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots \tag{1}$$

means that we can set

$$\cos\theta = \frac{e^{-i\theta} + e^{i\theta}}{2} \tag{2}$$

$$\sin\theta = \frac{i(e^{-i\theta} - e^{i\theta})}{2} \tag{3}$$

so that

$$\cos^2\theta + \sin^2\theta = 1.$$

An alternative definition of $\sin\theta$ as the negative of (3) changes the handedness of the coordinate system.

A function is *periodic* if

$$F(\theta + u) = F(\theta)$$

for some u. So for example $F(\theta + 2u) = F(\theta + u) = F(\theta)$.

An example of a periodic function is

$$G(\theta) = e^{im\theta},$$

where

$$G(\theta + 2\pi) = G(\theta).$$

Suppose we wish to describe the state of an arbitrary periodic complex function by

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} a_m e^{im\theta}, \; m \in \mathbb{M}_t, \tag{4}$$

where Ω is ordinal infinity in $\mathbb{M}_t$ (the summation then means over all $\pm m \in \mathbb{M}_t$) and the a_m are complex. We see this is periodic with period 2π and this period can be changed by adjusting the values of the $e^{im\theta}$ to $e^{im\theta/L}$ for L real. We suspend discussion of the convergence of (4) until later. Using (2) and (3) an alternative description is

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} (b_m \cos m\theta + c_m \sin m\theta) \tag{5}$$

for complex b_m and c_m. Since an arbitrary function f on $[\pi, -\pi]$ can be represented in terms of the even parity function

$$g(\theta) = \frac{f(\theta) + f(-\theta)}{2}$$

and the odd parity function

$$h(\theta) = \frac{f(\theta) - f(-\theta)}{2},$$

and the cos function is even and sin is odd, we can write this as

$$f(\theta) = \sum_{m=0}^{\Omega} B_m \cos m\theta + \sum_{m=1}^{\Omega} C_m \sin m\theta \qquad (6)$$

for complex B_m and C_m. The expansions (4) and (6) are known as *Fourier series*.

If f is an integrable function on the interval [a, b] of length L, so L = b – a, then formally we define the Fourier series of f for period L as

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} a_m e^{2\pi i m\theta/L}, \; m \in \mathbb{M}_t. \qquad (7)$$

We now observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}d\theta = 0 \text{ if } n \neq m, \text{ and } 1 \text{ if } n = m \qquad (8)$$

so we find for period $L = 2\pi$

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}d\theta. \qquad (9)$$

The quantity a_m is called the m^{th} *Fourier coefficient* of f.

The m^{th} Fourier coefficient of f for period L is then

$$a_m = \frac{1}{L}\int_{a}^{b} f(\theta)e^{-2\pi i m\theta/L}d\theta. \qquad (10)$$

Let $N \in \mathbb{M}_t$. We define the N^{th} *partial sum* of a Fourier series

$$S(N, \theta) = \sum_{m=-N}^{N} a_m e^{2\pi i m\theta/L}, \qquad (11)$$

so that $S(\Omega, \theta)$ is the Fourier series of $f(\theta)$, thought of as its limiting value.

We define the N^{th} *Dirichlet kernel* as

$$D_N(\theta) = \sum_{m=-N}^{N} e^{im\theta}, \qquad (12)$$

so this is obtained by setting $a_m = 1$ in the partial sum for period 2π.

Theorem 3.2.1. *The Dirichlet kernel has value*

$$\frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}.$$

Proof. Let $\omega = e^{i\theta}$. Equation (12) can be written as the sum of two geometric series in ω.

$$D_N(\theta) = \sum_{m=0}^{N} \omega^m + \sum_{m=-N}^{-1} \omega^m,$$

so using the formula for sums of geometric series this is

$$\frac{1 - \omega^{N+1}}{1 - \omega} - \frac{1 - \omega^{-N}}{1 - \omega} = \frac{\omega^{-N} - \omega^{N+1}}{1 - \omega} = \frac{\omega^{-N-1/2} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}},$$

which by the definition of the sine function in equation (3), is just

$$\frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}. \; \square$$

3.3. Convolutions.

The notion of a convolution is an important feature of Fourier analysis, and we will use it in later developments. Roughly speaking, convolutions are “weighted averages”.

In what follows we change the variable of integration and adopt the standard notation

$$\hat{f}(m) = a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}d\mu. \qquad (1)$$

The partial sum of equation 10.2.(11) with L = 2π now satisfies

$$\begin{aligned} S(N,\theta) &= \sum_{m=-N}^{N} \hat{f}(m)e^{im\theta} \\ &= \sum_{m=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}d\mu\right) e^{im\theta} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)\left(\sum_{m=-N}^{N} e^{im(\theta-\mu)}\right)d\mu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)\left(D_N(\theta-\mu)\right)d\mu. \ \square \end{aligned} \qquad (2)$$

To generalise equation (2) we introduce the convolution.

Definition 3.3.1. *Given two* 2π*-periodic continuous real functions* f *and* g*, the convolution of* $f * g$ *on* $[-\pi, \pi]$ *is*

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)g(\theta - \mu)d\mu. \qquad (3)$$

Equation (2) for the partial sum may now be represented using the convolution of f and the Dirichlet kernel D_N as

$$S(N,\theta) = (f * D_N)(\theta). \ \square \qquad (4)$$

Theorem 3.3.2. *For three* 2π*-periodic continuous real functions* f, g *and* h*, the convolution on* $[-\pi, \pi]$ *satisfies*

(i) $f * (g + h) = (f * g) + (f * h)$,
(ii) For any complex c, $(cf) * g = f * (cg)$,
(iii) $f * g = g * f$,
(iv) $f * (g * h) = (f * g) * h$,
(v) $f * g$ is continuous,
(vi) $\widehat{f * g} = \hat{f} * \hat{g}$.

Proof. Properties (i) and (ii) follow from the linearity of the convolution integral. □

Property (iii) on commutativity follows because the functions f and g are both periodic over the same interval. We can see if we interchange f and g that

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\mu)f(\theta - \mu)d\mu = (g * f)(\theta), \qquad (5)$$

since if $F(\mu) = f(\mu)g(\theta - \mu)$is continuous and periodic in this interval then

$$\int_{-\pi}^{\pi} F(\mu)d\mu = \int_{-\pi}^{\pi} F(\theta - \mu)d\mu \qquad (6)$$

for any real θ. Then the result follows on changing variables μ → -μ which reverses the order of integration but leaves the integral fixed, and translating μ → μ – θ which keeps the periodic nature of F. □

The proof of associativity in (iv) depends on swapping two integrals, together with changes of variables like (6). □

To prove (v), that $f * g$ is continuous, if f differs from $f + \delta$ by an infinitesimal and g differs from $g + \delta'$ by an infinitesimal, then from (i) and by the Eudoxus nature of $f * g$,

$$(f + \delta) * (g + \delta') = f * g + f * \delta' + \delta * g + \delta * \delta'$$

differs from it by an infinitesimal. □

Proposition (vi) is proved by evaluating

$$\begin{aligned} \widehat{f * g}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f * g)(\theta)e^{-im\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)g(\theta - \mu)d\mu\, e^{-im\theta}d\theta \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta - \mu)e^{-im(\theta - \mu)}\, d\theta\right) d\mu$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-im\theta}\, d\theta\right) d\mu$$
$$= \hat{f}(m) * \hat{g}(m). \square$$

Remark 3.3.3. Theorem 11.3.2 holds not only for f, g and h continuous, but also when they are integrable but not necessarily continuous. It uses the following lemma.

Lemma 3.3.4. *Let* f *be an either Eudoxus or infinitesimal integrable function on the circle. Then there exists a sequence of continuous functions* f_k *on the circle so that* $|f_k|$ *is Eudoxus or infinitesimal for all* k *and*

$$\int_{-\pi}^{\pi} |f(\mu) - f_\Omega(\mu)|\, d\mu$$

is infinitesimal or zero.

For a proof, see the appendix of [SS03]. This condition is important because we will deal with discrete structures and we need to know that the mathematics we will be applying is appropriate for them.

3.4. Good kernels, the Cesàro mean and Fejér's theorem.

Good kernels are a type of infinitely peaked weight distribution on a circle. We define the three properties of good kernels and give their meaning. Using convolutions we then show how these kernels can be employed to recover a specified function. Dirichlet kernels are not good kernels, but we will see that the Fegér kernels derived from them are good kernels.

Definition 3.4.1. If it exists, the *limit* L, with no infinitesimal part, of a function f is found from the value of f, say as a series, whenever (L – f) is infinitesimal or zero.

Definition 3.4.2. A family of kernels $\{K_m(\mu)\}_{m=1}^{\Omega}$ on the circle are *good kernels* whenever they satisfy the properties:

(i) For every m > 0,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_m(\mu)d\mu = 1.$$

This property says the kernel has unit mass around the circle.

(ii) For every m > 0 and absolute value $|K_m(\mu)|$ there exists an N > 0 satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_m(\mu)|d\mu \leq N,$$

so when $K_m(\mu) > 0$, property (ii) reduces to (i).

(iii) The *Dirac* δ *function* property holds. For every Eudoxus δ > 0

$$\int_{\delta \leq |\mu|}^{\pi} |K_\Omega(\mu)|d\mu \text{ is infinitesimal.}$$

Good kernels are related to convolutions as follows.

Theorem 3.4.3. (**good kernels as identity approximations**). *Let* $\{K_m(\mu)\}_{m=1}^{\Omega}$ *be a family of good kernels. Let* f *be an integrable function continuous at* μ. *Then* $(f * K_\Omega)(\mu)$ *differs from* $f(\mu)$ *by at most an infinitesimal.*

What we are implying above is that the convolution

$$(f * K_m)(\mu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - \mu)K_m(\mu)\, d\mu$$

has an average of $f(\theta - \mu)$ weighted by $K_m(\mu)$, and in the limit $m \to \boldsymbol{\Omega}$ its mass is infinitely concentrated at $\mu = 0$.

Proof. Let δ > 0 be Eudoxus. Then by property (i) of good kernels

$$(f * K_\Omega)(\mu) - f(\mu) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - \mu)K_\Omega(\mu)\, d\mu\right] - f(\mu)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta - \mu) - f(\mu)]K_\Omega(\mu)\, d\mu,$$

so that

$$|(f * K_\Omega)(\mu) - f(\mu)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu,$$
$$= \frac{1}{2\pi}\int_{|\mu|<\delta}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu$$
$$+ \frac{1}{2\pi}\int_{\delta\leq|\mu|\leq\pi}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu. \qquad (1)$$

By lemma 10.3.4 and properties (ii) and (iii) of good kernels these two terms on the right of (1) are at most infinitesimals. □

Under arbitrary bracketing an infinite series of complex numbers $s = \sum_{k=0}^{\Omega-1} c_k$, where $c_k \in \mathbb{C}$, may have ambiguous values. We will assume it is evaluated unambiguously as

$$s = (...((c_0 + c_1) + c_2) + ...$$

We define the N^th partial sum as the finite series

$$s_N = \sum_{k=0}^{N-1} c_k.$$

If we now look at the infinite series

$$s = 1 - 1 + 1 - 1 + ..., \qquad (2)$$

its partial sums are the ordered set, or sequence, {1, 0, 1, 0, ...}, so that we can form its average

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

In general this average value is known as the N^th *Cesàro mean* of s. The limit as $N \to \boldsymbol{\Omega}$ is the *Cesàro sum*, σ.

As can be seen from example (2) where $\sigma = \frac{1}{2}$, Cesàro summability can be applied to a wider class of cases than convergent series.

We will now apply Cesàro summability to Fourier series, forming the N^th *Fejér kernel* $F_N(\theta)$ as a Cesàro mean of Dirichlet kernels.

$$F_N(\theta) = \frac{D_0(\theta) + D_1(\theta) + \cdots + D_{N-1}(\theta)}{N}.$$

Theorem 3.4.4. *The Fejér kernel has value*

$$F_N(\theta) = \frac{1}{N}\frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)}.$$

Proof. We have seen in theorem 10.2.1 that putting $\omega = e^{i\theta}$ the Dirichlet kernel has value

$$D_N(\theta) = \frac{\omega^{-N} - \omega^{N+1}}{1 - \omega}.$$

Thus

$$D_0(\theta) + D_1(\theta) + \cdots + D_{N-1}(\theta) = \frac{\omega^0 - \omega^1 + \omega^{-1} - \omega^2 + ... + \omega^{-N} - \omega^{N+1}}{1 - \omega}, \qquad (3)$$

so using

$$\frac{(1-\omega)^2}{\omega} = \omega^{-1} + \omega - 2,$$

and on multiplying the numerator and denominator of (3) by $(1 - \omega)$, from the definition of the sine function in 7.2.(3) we get

$$F_N(\theta) = \frac{1}{N}\frac{\omega^{-N-1} + \omega^{N+1} - 2}{\omega^{-1} + \omega - 2} = \frac{1}{N}\frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)}. \square \tag{4}$$

Theorem 3.10.5. *The Fejér kernel is a good kernel.*

Proof. Properties (i), (ii) and (iii) of definition 3.10.2 are obtained from the properties of the Dirichlet kernel, also (ii) from the fact that (4) is positive and (iii) that $1/N\sin^2(\theta/2)$ is an infinitesimal for N = $\boldsymbol{\Omega}$ and $0 < \theta \leq \pi$. □

3.5. The Maurer-Cartan equation (~).

3.6. The intricate analytic Cauchy-Riemann equations.

Suppose that

$$P(g) = w(g) + x(g)i + y(g)\alpha + z(g)\phi \tag{1}$$

is a function of an intricate number, g ∊ Я. The *intricate derivative* of P at a point g_0 is defined by

$$\lim_{\substack{h \to 0 \\ h \in Я}} \frac{P(g_0 + h) - P(g_0)}{h}, \tag{2}$$

wherever this limit is path independent of h to 0. Alternatively, to obtain this derivative, we can regard h as a (hyper)infinitesimal number and discard (hyper)infinitesimal parts of the division performed in (2). Note that when an intricate numerator is divided by the intricate denominator h, the result depends on whether the reciprocal h^{-1} is multiplied first or last, so we assume two types of derivative. When h comes first we write $\lim_{h \to 0}$, otherwise the limit is from the right, denoted by $\lim_{0 \leftarrow h}$.

A standard argument indicates for this left limit to be unique, intricate polynomials satisfy the condition of being intricate left analytic, that is we have

$$P(w, x, y, z) = P(w + xi + y\alpha + z\phi, 0, 0, 0). \tag{3}$$

If the limit (2) exists, it can be computed by taking the limit h → 0 along the real, imaginary, actual or phantom axes. In all cases it should give the same result. Approaching along the real axis, we find for the limit on the left

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h) - P(g_0)}{h} = \frac{\partial P(g_0)}{\partial w} \quad , \tag{4}$$

whereas along the imaginary axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + hi) - P(g_0)}{hi} = \frac{1}{i}\frac{\partial P(g_0)}{\partial x}, \tag{5}$$

along the actual axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h\alpha) - P(g_0)}{h\alpha} = \frac{1}{\alpha}\frac{\partial P(g_0)}{\partial y}, \tag{6}$$

and for the phantom axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h\phi) - P(g_0)}{h\phi} = \frac{1}{\phi}\frac{\partial P(g_0)}{\partial z}. \tag{7}$$

Then to be intricate left analytic the equality of the results (4) to (7) gives

$$\frac{\partial P}{\partial x} = i\frac{\partial P}{\partial w} \tag{8}$$

$$\frac{\partial P}{\partial y} = \alpha\frac{\partial P}{\partial w} \quad (9)$$
$$\frac{\partial P}{\partial z} = \phi\frac{\partial P}{\partial w}, \quad (10)$$
which are the 3 left intricate Cauchy–Riemann equations at the point g_0. ☐

If we change the basis by means of continuous first derivatives to P = r + si + tα + uϕ, then

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$
$$\frac{\partial P}{\partial x} = \frac{\partial r}{\partial x} + i\frac{\partial s}{\partial x} + \alpha\frac{\partial t}{\partial x} + \phi\frac{\partial u}{\partial x}$$
$$\frac{\partial P}{\partial y} = \frac{\partial r}{\partial y} + i\frac{\partial s}{\partial y} + \alpha\frac{\partial t}{\partial y} + \phi\frac{\partial u}{\partial y}$$
$$\frac{\partial P}{\partial z} = \frac{\partial r}{\partial z} + i\frac{\partial s}{\partial z} + \alpha\frac{\partial t}{\partial z} + \phi\frac{\partial u}{\partial z},$$

giving, using equations (8) to (10), on multiplying on the left

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$
$$-\frac{\partial P}{\partial w} = i\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} - \phi\frac{\partial t}{\partial x} + \alpha\frac{\partial u}{\partial x}$$
$$\frac{\partial P}{\partial w} = \alpha\frac{\partial r}{\partial y} + \phi\frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} + i\frac{\partial u}{\partial y}$$
$$\frac{\partial P}{\partial w} = \phi\frac{\partial r}{\partial z} - \alpha\frac{\partial s}{\partial z} - i\frac{\partial t}{\partial z} + \frac{\partial u}{\partial z},$$

so that the intricate Cauchy-Riemann equations become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \quad (11)$$
$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = \frac{\partial u}{\partial y} = -\frac{\partial t}{\partial z} \quad (12)$$
$$\frac{\partial t}{\partial w} = -\frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = -\frac{\partial s}{\partial z} \quad (13)$$
$$\frac{\partial u}{\partial w} = \frac{\partial t}{\partial x} = \frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \quad (14)$$

the number of these latter equations being 16.

We represent the Jacobian matrix of equations (11) to (14) by

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{bmatrix}, \quad (15)$$

or as the 2-hyperintricate

$$a1_1 + b1_i + c\phi_\alpha + d\phi_\phi. \quad (16)$$

Conversely, if P : Я → Я is a differentiable function when regarded as a function on $\mathbb{U}^4$, then P is a left intricate-valued real-differentiable function if and only if the 3 left intricate-differential Cauchy-Riemann equations hold. ☐

From equation (11)

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x},$$

then if the second differential exists

$$\frac{\partial^2 r}{\partial w^2} = \frac{\partial^2 s}{\partial w \partial x},$$

and from equation (12)

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x},$$

so

$$-\frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 s}{\partial w \partial x},$$

giving

$$\frac{\partial^2 r}{\partial w^2} + \frac{\partial^2 r}{\partial x^2} = 0. \quad (17)$$

Further relations obtained in a similar way are

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial y^2} = 0, \quad (18)$$

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial z^2} = 0, \quad (19)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial x^2} = 0, \quad (20)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial y^2} = 0, \quad (21)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial z^2} = 0, \quad (22)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial x^2} = 0, \quad (23)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial y^2} = 0, \quad (24)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial z^2} = 0, \quad (25)$$

$$\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad (26)$$

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (27)$$

and

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial z^2} = 0. \ \square \quad (28)$$

A function with continuous second derivative satisfying (17) or (26) is said to be *harmonic*.

For the right limit to be unique, intricate polynomials P' satisfy the condition of being intricate right analytic, that is by taking the limit 0 ← h along the real, imaginary, actual or phantom axes, on approaching along the real axis we find for the limit on the right

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h) - P'(g_0)}{h} = \frac{\partial P'(g_0)}{\partial w}, \quad (29)$$

whereas along the imaginary axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + hi) - P'(g_0)}{hi} = \frac{\partial P'(g_0)}{\partial x}\frac{1}{i}, \quad (30)$$

along the actual axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\alpha) - P'(g_0)}{h\alpha} = \frac{\partial P'(g_0)}{\partial y}\frac{1}{\alpha}, \quad (31)$$

and for the phantom axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\phi) - P'(g_0)}{h\phi} = \frac{\partial P'(g_0)}{\partial z}\frac{1}{\phi}. \quad (32)$$

Then to be intricate right analytic the equality of the results (29) to (32) gives

$$\frac{\partial P'}{\partial x} = \frac{\partial P'}{\partial w}i \quad (33)$$

$$\frac{\partial P'}{\partial y} = \frac{\partial P'}{\partial w}\alpha \quad (34)$$

$$\frac{\partial P'}{\partial z} = \frac{\partial P'}{\partial w}\phi, \quad (35)$$

which are the 3 right intricate Cauchy–Riemann equations at the point g_0. □

If we multiply on the right we obtain

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$
$$-\frac{\partial P}{\partial w} = i\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} + \phi\frac{\partial t}{\partial x} - \alpha\frac{\partial u}{\partial x}$$
$$\frac{\partial P}{\partial w} = \alpha\frac{\partial r}{\partial y} - \phi\frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} - i\frac{\partial u}{\partial y}$$
$$\frac{\partial P}{\partial w} = \phi\frac{\partial r}{\partial z} + \alpha\frac{\partial s}{\partial z} + i\frac{\partial t}{\partial z} + \frac{\partial u}{\partial z},$$

so that the intricate Cauchy-Riemann equations now become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \quad (36)$$
$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial t}{\partial z} \quad (37)$$
$$\frac{\partial t}{\partial w} = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = \frac{\partial s}{\partial z} \quad (38)$$
$$\frac{\partial u}{\partial w} = -\frac{\partial t}{\partial x} = -\frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \quad (39)$$

We represent the Jacobian matrix of equations (36) to (39) as

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}, \quad (40)$$

where the 2-hyperintricate

$$a1_1 + b\alpha_i + c\phi_1 + di_i, \quad (41)$$

has been obtained from equation (40). □

More generally, an intricate polynomial P″ may be allocated any combination of the following intricate mixed analytic expressions

$$\frac{\partial P''}{\partial x} = i\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}i, \quad (42)$$
$$\frac{\partial P''}{\partial y} = \alpha\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}\alpha, \quad (43)$$
$$\frac{\partial P''}{\partial z} = \phi\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}\phi. \quad (44)$$

Thus the structure of these transformations differs depending on whether multiplication occurs on the left or on the right.□

3.7. J-diffeomorphisms and non-fixed J.

P may be expressed for intricate numbers in J format as

$$P = r + Jv, \quad (1)$$

It may or may not be the case that J is constant. Chapter 9 uses the fact that for intricate numbers constant J implies J-abelian.

When a diffeomorphism is applied to $J = si + t\alpha + u\phi \to J + \delta J$ so that $J^2 = (J + \delta J)^2$, where $J^2 = 0$ or ± 1, then

$$s = t(\partial t/\partial s) + u(\partial u/\partial s). \text{ □} \quad (2)$$

3.8. Intricate components and relative orientation.

3.9. The intricate Cauchy-Riemann equations are nonconformal.

If we look at the Cauchy–Riemann equations 10.5.(11) to (14) written in 10.5.(15) as a 2-hyperintricate Jacobian matrix, and likewise for the Jacobian matrix 10.5.(40), a matrix of this form contains within it components of non-complex type (is not hyperimaginary), that is the matrix is not composed of layers containing only 1 and i. Geometrically, multiplying a complex number by a complex number is always the composition of a rotation with a scaling, and in particular preserves angles, but a non-complex matrix does not. Expressed more simply, for a complex number

$$e^{\rho}e^{i\theta}.e^{i\theta'} = e^{\rho}e^{i(\theta + \theta')} = e^{\rho + i(\theta + \theta')}, \qquad (1)$$

and this is the core idea of what we are saying. For a general intricate number (see *Superexponential algebra*, chapter XV, section 3), the intricate $e^{p + (bi + c\alpha + d\phi)K}$ cannot usually be expressed as $e^{w}e^{xi}e^{y\alpha}e^{z\phi}$ with p = w, bK = x, cK = y and dK = z.

However, if two (hyper-)intricate numbers are J-abelian with the same $J^2 = -1$, then

$$e^{\rho}e^{J\theta}.e^{J\theta'} = e^{\rho}e^{J(\theta + \theta')} = e^{\rho + J(\theta + \theta')}, \qquad (2)$$

and the transformation is now conformal.

Consequently, a function satisfying the intricate Cauchy–Riemann equations, with a nonzero derivative does not in general preserve the angle between curves in the plane, so we say the intricate Cauchy–Riemann equations contain already the property for a function not to be conformal. ☐

3.10. The zargonion analytic Cauchy-Riemann equations.

We wish to have a symbol designed for putting commas between ordered sets of coefficients of hyperintricate basis elements. For $0 < k \leq m \in \mathbb{N}$ define

$$(,)_{k=1}^{m} w_k$$

to be (w_1) when m = 1, (w_1, w_2) when m = 2 and to have the properties for $m' \leq m$

$$(,)_{k'=1}^{m'} w_{k'} = (,)_{k=1}^{m'-k'+1} w_{k'}, \qquad (1)$$

$$(,)_{k=1}^{m} w_k = [(,)_{k=1}^{m'} w_k][(,)_{m'+1}^{m} w_k]. \qquad (2)$$

Let an n-hyperintricate number be denoted by $Я_n$, with coefficients of basis elements w_k. We define the hyperintricate analytic condition for a function P corresponding to 10.6.(3) to be

$$P[(,)_{k=1}^{4^n} w_k] = P[Я_n, (,)_{k=1}^{4^n-1} 0]. \qquad (3)$$

The derivative of hyperintricate numbers satisfies the same equation as 10.6.(2). Then if γ_j is a basis element with coefficient $w_{\gamma j}$, the equations corresponding to 10.6.(8)-(10) are

$$\frac{\partial P}{\partial w} = \gamma_j^{-1}\frac{\partial P}{\partial w_{\gamma j}}. \qquad (4)$$

Set

$$P = \sum_{k=1}^{4^n} \gamma_k r_k, \qquad (5)$$

so that if we change basis then

$$\frac{\partial P}{\partial w_{\gamma j}} = \sum_{k=1}^{4^n} \gamma_k \frac{\partial r_k}{\partial w_{\gamma j}}, \qquad (6)$$

and multiplying this on the left by γ_j^{-1}, using (4)

$$\frac{\partial P}{\partial w} = \gamma_j^{-1} \sum_{k=1}^{4^n} \gamma_k \frac{\partial r_k}{\partial w_{\gamma j}}. \quad (7)$$

For multiplying on the left, let $<\gamma_j^{-1}, \gamma_k>_L$ equal the sign of $\gamma_j^{-1}\gamma_k$ in that order when $\gamma_j^{-1} = \pm\gamma_k$, and otherwise be 0, so that the hyperintricate left derivative Cauchy-Riemann equations become, on equating hyperintricate parts

$$<\gamma_{j'}^{-1}, \gamma_k>_L[<\gamma_j^{-1}, \gamma_k>_L\frac{\partial r_k}{\partial w_{\gamma j}}] = <\gamma_{j'}^{-1}, \gamma_k>_L[<\gamma_{j'}^{-1}, \gamma_{k'}>_L\frac{\partial r_{k'}}{\partial w_{\gamma j'}}]. \quad (8)$$

For multiplying on the right, a similar situation obtains, on defining $<\gamma_j^{-1}, \gamma_k>_R$ equal to the sign of $\gamma_k\gamma_j^{-1}$ (in reversed order from previously) when $\gamma_j^{-1} = \pm\gamma_k$, otherwise 0. The right derivative hyperintricate Cauchy-Riemann equations are obtained by substituting $< , >_L$ in (8) with $< , >_R$. □

As we have seen, the quaternions are an instance of hyperintricate numbers, to which these considerations apply. The proof that quaternion analytic Cauchy-Riemann equations are conformal and the extension of this idea to the octonions and n-novanions is now discussed.

3.11. The complex Cauchy integral formula (~).

Holomorphic functions are the central objects of study in complex analysis. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of every point in its domain. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series.

The term analytic function is often used interchangeably with holomorphic function, although the word "analytic" is also used in a broader sense to describe any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

Holomorphic functions are also sometimes referred to as conformal maps. A holomorphic function whose domain is the whole complex plane is called an entire function. The phrase "holomorphic at a point z_0" means not just differentiable at z_0, but differentiable everywhere within some neighbourhood of z_0 in the complex plane.

3.12. The zargonion Cauchy integral formula (~).

3.13. Picard's theorems (~).

3.14. Singularities of mixed twisted and untwisted structures.

For an intricate number

$$H = a1 + bi + c\alpha + d\phi$$

this can be represented as

$$H = (a1 + bi) + \alpha(c1 + di).$$

For hyperimaginary numbers an example might be

$$H' = (a1_1 + bi_i) + (c1_i + di_1),$$

where we have collected together with terms in the first brackets with an even number of i's in their hyperintricate representation, and the second brackets containing an odd number of i's. We might now expand this example to contain a general hyperintricate number, for example

$$H'' = [a1_1 + b\alpha_\alpha](c1_1 + di_i) + [e1_1 + f\alpha_\alpha](g1_i + hi_1) + [j\alpha_1 + k1_\alpha](m1_1 + ni_i) + [p\alpha_1 + q1_\alpha](r1_i + si_1). \qquad (1)$$

We will use general representations similar to type (1) to describe conformal components of Cauchy-Riemann type and corresponding nonconformal components.

3.15. Nonconformal structures are mixed and singular.