CHAPTER 5

The Riemann-Roch theorem.

5.1. Introduction.

5.2. The Riemann-Roch theorem.

5.3. Diffeomorphisms.

5.4. Bézout's theorem.

5.5. The Gauss-Bonnet theorem.

Shorn of sophisticated description, a Riemannian manifold is an n-dimensional space given by a local quadratic metric

$$\sum_{j,k} g_{jk} dx_j dx_k = g_{00}dx_0^2 + g_{01}dx_0dx_1 + g_{10}dx_1dx_0 + ...$$

The negative derivative of the unit vector field which is normal, that is, at right angles, to a surface is called the shape operator (or Weingarten map or second fundamental tensor). The shape operator is a curvature of a submanifold of a manifold which depends on its particular embedding, also called the extrinsic curvature, and the Gaussian curvature is given by the determinant of it, and does not depend on the embedding.

For a two-dimensional Riemannian manifold without boundary, the integral of the Gaussian curvature over the entire manifold with respect to area is 2π times the Euler characteristic of the manifold. This is the Gauss-Bonnet theorem. It is interesting that the total Gaussian curvature is differential-geometric in character, but the Euler characteristic is topological and does not depend on differential geometry. So if we distort the surface and change the curvature at any location, regardless of how we do it, the same total curvature is maintained.

5.6. The Atiyah-Singer index theorem.

5.7. Embeddings and projections.

Chapter XVI defines a superexponential variety. Reduced to the case of a matrix variety, a typical example could be

$$aX^3Y + bX^2YZ^5 + cZ = 0,$$

where X, Y and Z are matrices and a, b, c are coefficients. In this section we will deal with quadratic forms where the matrices and coefficients are reduced to complex variables.

Chapter XI, section 2, proved Sylvester's law of inertia, which states that the signature, or difference between positive and negative terms, derived from the quadratic form $\sum x_j b_{jk} x_k$, where b_{jk} is symmetric, is invariant under any similarity transformation to

$$x_1^2 + ... + x_r^2 - x_{r+1}^2 - ... - x_n^2. \qquad (1)$$

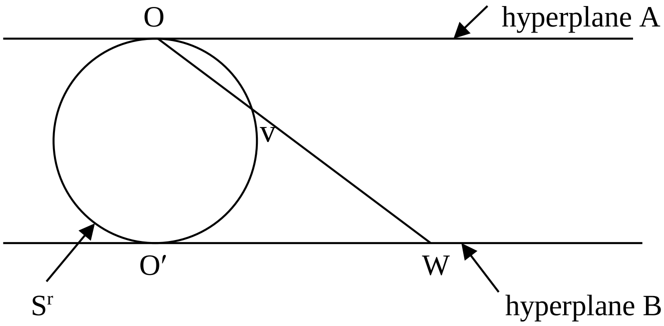
We have already seen such forms in the norm squared of an intricate number, and as an everywhere positive squared norm for a quaternion, octonion, or otherwise n-novanion. The determinant can be defined from the norm squared, but from the n-hyperintricate number $Я_n$

multiplied by its conjugate $Я_n$* to form det$Я_n$, then if the norm squared is quadratic and n > 1 there is cancellation of common factors between $Я_n$* and det$Я_n$.

A classical case we wish to consider is stereographic projection from a sphere to a complex plane, which we will generalise to an r-sphere mapped to an r-hyperplane. A sphere has n = r in equation (1) above, and we wish to generalise this to stereographic projection when n > r, so this corresponds to hyperbolic geometry, and we will look at this.

Let S^r be the r-sphere. The r-component of numbers with quadratic norms of the form (1), represented by some points on S^r, may be mapped to the tangent hyperplanes A and B below. Let the stereographic map from B, at W in the diagram, to S^r be p^{-1}.

We extend the stereographic projection map p to the mapping including the punctured point O in S^r to the codomain given by hyperplane A, tangent at O. This hyperplane belongs to an *indeterminate zero algebra* [Ad14], chapter II, defined by 0/0 and which differs from the usual extension to a 'point at infinity'. Alternatively O is at c0 for a *determinate zero algebra* and cЧ is ultrainfinity.



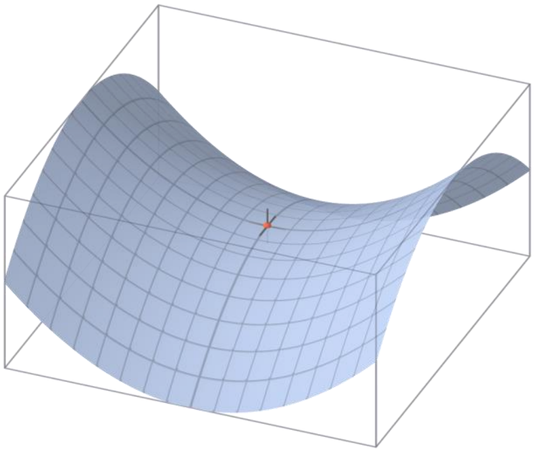
Near O, an infinitesimal h $\not\in$ p^{-1}(A), and h $\not\in$ p^{-1}(B) provided all elements of B < 1/h, but if B contains ordinal infinities, h ∊ p^{-1}(B). ☐

The flat hyperplane B may be replaced by a hyperboloid with cross section

$$x_1^2 - x_{r+1}^2 = \pm R$$

to give a more general mapping. ☐

[Wikipedia] *Hyperbolic space* is a homogeneous space that has a constant negative curvature. It is hyperbolic geometry in more than 2 dimensions, and is distinguished from Euclidean spaces with zero curvature that defines the Euclidean geometry, and elliptic geometry with a constant positive curvature.



When embedded inside a Euclidean space (of a higher dimension), every point of a hyperbolic space is a saddle point. Another distinctive property is the amount of space covered by the n-ball in hyperbolic n-space: it increases exponentially with respect to the radius of the ball for large radii, rather than as a polynomial.

Hyperbolic n-space, denoted $\mathbb{H}^n$, is a maximally symmetric, simply connected, n-dimensional Riemannian manifold with a constant negative curvature. Hyperbolic space is a space exhibiting hyperbolic geometry. It is the negative-curvature analogue of the n-sphere. Hyperbolic 2-space, $\mathbb{H}^2$, is also called the hyperbolic plane. Although hyperbolic space $\mathbb{H}^n$ is diffeomorphic to $\mathbb{U}^n$, its negative-curvature metric gives it very different geometric properties.

We will now discuss models of hyperbolic space.

Hyperbolic space, developed independently by Nikolai Lobachevsky and János Bolyai, is a geometrical space analogous to Euclidean space, but such that Euclid's parallel postulate is no longer assumed to hold. Instead, the parallel postulate is replaced by the following alternative in two dimensions:

- Given any line L and point P not on L, there are at least two distinct lines passing through P which do not intersect L.

It is then a theorem that there are infinitely many such lines through P. This axiom still does not uniquely characterise the hyperbolic plane up to isometry. There is an extra constant, the curvature $K < 0$, which must be specified. However, it does uniquely characterise it up to bijections which only change the notion of distance by an overall constant, called *homothety*. By choosing an appropriate length scale, we can thus assume, without loss of generality, that $K = -1$.

Models of hyperbolic spaces that can be embedded in a flat, for example Euclidean, space may be constructed. In particular, the existence of model spaces implies that the parallel postulate is logically independent of the other axioms of Euclidean geometry.

There are several important models of hyperbolic space: the *Klein model*, the *hyperboloid model*, the *Poincaré ball model* and the *Poincaré half space model*. These all model the same geometry in the sense that any two of them can be related by a transformation that preserves all the geometrical properties of the space, including isometry (though not with respect to the metric of a Euclidean embedding). An isometry is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space.

(i) The hyperboloid model.

The hyperboloid model realises hyperbolic space as a hyperboloid in $\mathbb{U}^{n+1} = \{(x_0, ..., x_n): x_i \in \mathbb{U}, i = 0, 1, ..., n\}$. The hyperboloid is the locus $\mathbb{H}^n$ of points whose coordinates satisfy

$$x_0^2 - x_1^2 - ... - x_n^2 = 1, x_0 > 0.$$

In this model a *line* (or geodesic) is the curve formed by the intersection of $\mathbb{H}^n$ with a plane through the origin in $\mathbb{U}^{n+1}$.

The hyperboloid model is closely related to the geometry of Minkowski space. The quadratic form

$$Q(x) = x_0^2 - x_1^2 - ... - x_n^2,$$

which defines the hyperboloid, polarises to give the bilinear form

$$B(x, y) = [Q(x + y) - Q(x) - Q(y)]/2$$
$$= x_0y_0 - x_1y_1 - ... - x_ny_n,$$

The space $\mathbb{U}^{n+1}$, equipped with the bilinear form B, is an (n + 1)-dimensional Minkowski space $\mathbb{U}^{n,1}$.

One can associate a *distance* on the hyperboloid model by defining the distance between two points x and y on $\mathbb{H}$ to be

$$d(x, y) = \text{arcosh } B(x, y).$$

This function satisfies the axioms of a metric space. It is preserved by the action of the Lorentz group on $\mathbb{U}^{n,1}$. Hence the Lorentz group acts as a transformation group preserving isometry on $\mathbb{H}^n$. ☐

(ii) The Klein model.

An alternative model of hyperbolic geometry is on a certain domain in projective space. The Minkowski quadratic form Q defines a subset $V^n \subset \mathbb{U}\mathbf{P}^n$ given as the locus of points for which $Q(x) > 0$ in the homogeneous coordinates x. The domain V^n is the *Klein model* of hyperbolic space.

The lines of this model are the open line segments of the ambient projective space which lie in V^n. The distance between two points x and y in V^n is defined by

$$d(x, y) = \text{arcosh}\left(\frac{B(x,y)}{\sqrt{Q(x)Q(y)}}\right).$$

This is well-defined on projective space, since the ratio under the inverse hyperbolic cosine is homogeneous of degree 0.

This model is related to the hyperboloid model as follows. Each point $x \in V^n$ corresponds to a line L_x through the origin in $\mathbb{U}^{n+1}$, by the definition of projective space. This line intersects the hyperboloid $\mathbb{H}^n$ in a unique point. Conversely, through any point on $\mathbb{H}^n$, there passes a unique line through the origin (which is a point in the projective space). This correspondence defines a bijection between V^n and $\mathbb{H}^n$. It is an isometry, since evaluating d(x,y) along Q(x) = Q(y) = 1 reproduces the definition of the distance given for the hyperboloid model. ☐

(iii) The Poincaré ball model.

A closely related pair of models of hyperbolic geometry are the Poincaré ball and Poincaré half-space models.

The ball model comes from a stereographic projection of the hyperboloid in $\mathbb{U}^{n+1}$ onto the hyperplane $\{x_0 = 0\}$. In detail, let S be the point in $\mathbb{U}^{n,1}$ with coordinates (−1, 0, 0, ..., 0): the *south pole* for the stereographic projection. For each point P on the hyperboloid $\mathbb{H}^n$, let P^* be the unique point of intersection of the line SP with the plane $\{x_0 = 0\}$.

This establishes a bijective mapping of $\mathbb{H}^n$ into the unit ball

$$B^n = \{(x_1, ..., x_n): x_1^2 + ... + x_n^2 < 1\}$$

in the plane $\{x_0 = 0\}$.

The geodesics in this model are semicircles that are perpendicular to the boundary sphere of B^n. Isometries of the ball are generated by spherical inversion in hyperspheres perpendicular to the boundary. ☐

(iv) The Poincaré half-space model.

The half-space model results from applying inversion in a circle with centre a boundary point of the Poincaré ball model B^n above and a radius of twice the radius.

This sends circles to circles and lines, and is moreover a conformal transformation. Consequently, the geodesics of the half-space model are lines and circles perpendicular to the boundary hyperplane. ☐

Considering hyperbolic manifolds, every complete, connected, simply connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space $\mathbb{H}^n$. As a result, the universal cover of any closed manifold M of constant negative curvature -1, which is to say, a hyperbolic manifold, is $\mathbb{H}^n$. Thus, every such M can be written as $\mathbb{H}^n/\Gamma$ where Γ is a finite discrete group of isometries on $\mathbb{H}^n$. That is, Γ is a lattice in $SO^+(n, 1)$.

For Riemann surfaces, two-dimensional hyperbolic surfaces can also be understood according to the language of Riemann surfaces. According to the uniformisation theorem, every Riemann surface is elliptic, parabolic or hyperbolic. Most hyperbolic surfaces have a non-trivial fundamental group $\pi_1 = \Gamma$, where the groups that arise this way are known as Fuchsian groups. The quotient space $\mathbb{H}^2/\Gamma$ of the upper half-plane modulo the fundamental group is known as the Fuchsian model of the hyperbolic surface. The Poincaré half plane is also hyperbolic, but is simply connected and noncompact. It is the universal cover of the other hyperbolic surfaces.

The analogous construction for three-dimensional hyperbolic surfaces is the Kleinian model.

5.8. The Lefschetz fixed point theorem.