CHAPTER 9

Local zeta functions

9.1. Introduction.

The *Weil conjectures* were some highly influential proposals by André Weil (1949) on the generating functions, known as local zeta functions, derived from counting the number of points on algebraic varieties over finite fields.

A variety V over a finite field with q elements has a finite number of rational points, as well as points over every finite field with q^k elements containing that field. The generating function has coefficients derived from the numbers N_k of points over the essentially unique field with q^k elements.

Weil conjectured that such *zeta-functions* should be rational functions, should satisfy a form of functional equation, and should have their zeros in restricted places. The last two parts were quite consciously modeled on the Riemann zeta function and Riemann hypothesis. The rationality was proved by Dwork (1960), the functional equation by Grothendieck (1965), and the analogue of the Riemann hypothesis was proved by Deligne (1974).

In this chapter we prove the Riemann conjecture for function fields (the Hasse-Weil bound). We can also express these ideas for zeta functions for corresponding L-series. The original proof of the Weil conjectures is given in what is known as SGA4½ [2De77], where the reader is expected to understand Hilbert's theorem 90, the Lefschetz fixed point theorem, Gauss, Jacobi and Kloosterman sums and the Brauer and Picard groups. SGA contains a description of étale cohomology, which is also given in the appendix of [1Ca85]. For a detailed alternative discussion, the reader is referred to [HKT08] for which we have already made available intersection theory, Bézout's theorem and the Riemann-Roch theorem.

9.2. Background and history of the Weil conjectures.

The earliest precursor of the Weil conjectures is by C.F. Gauss and appears in section VII of his *Disquisitiones Arithmetica,* concerned with roots of unity and Gaussian periods. In article 358, he moves on from the periods that build up towers of quadratic extensions, for the construction of regular polygons, and assumes that p is a prime number such that $p - 1$ is divisible by 3. Then there is a cyclic cubic field inside the cyclotomic field of pth roots of unity, and a normal integral basis of periods for the integers of this field, an instance of the Hilbert-Speiser theorem. Gauss constructs the order-3 periods, corresponding to the cyclic group $(\mathbf{Z}/p\mathbf{Z})^\times$ of non-zero residues modulo p under multiplication and its unique subgroup of index three. Gauss lets R, R′, and R″ be its cosets. Taking the periods, the sums of roots of unity, corresponding to these cosets applied to $\exp(2\pi i/p)$, he notes that these periods have a multiplication table that is accessible to calculation. Products are linear combinations of the periods, and he determines the coefficients. He sets, for example, (RR) equal to the number of elements of $\mathbf{Z}/p\mathbf{Z}$ which are in R and which, after being increased by one, are also in R. He proves that this number and related ones are the coefficients of the products of the periods.

To see the relation of these sets to the Weil conjectures, notice that if α and α + 1 are both in R, then there exist x and y in $\mathbf{Z}/p\mathbf{Z}$ such that $x^3 = \alpha$ and $y^3 = \alpha + 1$. Consequently, $x^3 + 1 = y^3$.

Therefore (RR) is the number of solutions to $x^3 + 1 = y^3$ in the finite field **Z**/p**Z**. The other coefficients have similar interpretations. Gauss's determination of the coefficients of the products of the periods therefore counts the number of points on these elliptic curves, and as a byproduct he proves the analogue of the Riemann hypothesis.

The Weil conjectures in the special case of algebraic curves were conjectured by E. Artin (1924). The case of curves over finite fields was proved by Weil, finishing the project started by Hasse's theorem on elliptic curves over finite fields. They implied upper bounds for exponential sums, a basic concern in analytic number theory (Moreno 2001).

What was really eye-catching, from the point of view of other mathematical areas, was the proposed connection with algebraic topology. Given that finite fields are *discrete* in nature, and topology speaks only about the *continuous*, the detailed formulation of Weil, based on working out some examples, was striking and novel. It suggested that geometry over finite fields should fit into well-known patterns relating to Betti numbers, the Lefschetz fixed-point theorem and so on.

The analogy with topology suggested that a new homological theory be set up applying within algebraic geometry. This took two decades. It was a central aim of the work and school of Alexander Grothendieck, building up on initial suggestions from J-P Serre. The rationality part of the conjectures was proved first by Bernard Dwork (1960), using p-adic methods. Grothendieck (1965) and his collaborators established the rationality conjecture, the functional equation and the link to Betti numbers by using the properties of étale cohomology, a new cohomology theory developed by Grothendieck and Artin for attacking the Weil conjectures, as outlined in Grothendieck (1960). Of the four conjectures the analogue of the Riemann hypothesis was the hardest to prove. Motivated by the proof of Serre (1960) of an analogue of the Weil conjectures for Kähler manifolds, Grothendieck envisioned a proof based on his standard conjectures on algebraic cycles (Kleiman 1968). However, Grothendieck's standard conjectures remain open, except for the hard Lefschetz theorem, which was proved by Deligne by extending his work on the Weil conjectures, and the analogue of the Riemann hypothesis was proved by Deligne (1974), using the étale cohomology theory but circumventing the use of standard conjectures by an ingenious argument.

Deligne (1980) found and proved a generalization of the Weil conjectures, bounding the weights of the pushforward of a sheaf.

9.3. The statement of the Weil conjectures.

Suppose that X is a non-singular n-dimensional projective algebraic variety over the field F_q with q elements. The *zeta function* $\zeta(X, s)$ of X is by definition

$$\zeta(X, s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-ms}\right).$$

The Weil conjectures state:

(1) *Rationality*. $\zeta(X, s)$ is a rational function of $T = q^{-s}$. More precisely, $\zeta(X, s)$ can be written as a finite alternating product

$$\prod_{i=0}^{2n} P_i\left(q^{-s}\right)^{-1^{i+1}} = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

where each $P_i(T)$ is an integral polynomial. Furthermore, $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^nT$, and for $1 \leq i \leq 2n - 1$, $P_i(T)$ factors over $\mathbb{C}$ as $\prod_j (1 - \alpha_{ij}T)$ for some numbers α_{ij}.

(2) *Functional equation and Poincaré duality*. The zeta function satisfies

$$\zeta(X, n - s) = \pm q^{\frac{nE}{2} - Es}\zeta(X, s)$$

or equivalently

$$\zeta(X, q^{-n}T^{-1}) = \pm q^{\frac{nE}{2}}T^E\zeta(X, T)$$

where E is the Euler characteristic of X. In particular, for each i, the numbers $\alpha_{2n-i,1}$, $\alpha_{2n-i,2}$, … equal the numbers $q^n/\alpha_{i,1}$, $q^n/\alpha_{i,2}$, … in some order.

(3) *Riemann hypothesis*. $|\alpha_{i,j}| = q^{i/2}$ for all $1 \leq i \leq 2n - 1$ and all j. This implies that all zeros of $P_k(T)$ lie on the 'critical line' of complex numbers s with real part k/2.

(4) *Betti numbers*. If X is a (good) 'reduction mod p' of a non-singular projective variety Y defined over a number field embedded in the field of complex numbers, then the degree of P_i is the i^{th} Betti number of the space of complex points of Y.

9.4. The projective line.

The simplest example (other than a point) is to take X to be the projective line. The number of points of X over a field with q^m elements is just $N_m = q^m + 1$ (where the '+ 1' comes from the 'point at infinity'). The zeta function is just

$$1/(1 - q^{-s})(1 - q^{1-s}).$$

It is easy to check all parts of the Weil conjectures directly. For example, the corresponding complex variety is the Riemann sphere and its initial Betti numbers are 1, 0, 1.

9.5. Projective space.

It is not much harder to do n dimensional projective space. The number of points of X over a field with q^m elements is just $N_m = 1 + q^m + q^{2m} + ... + q^{nm}$. The zeta function is just

$$1/(1 - q^{-s})(1 - q^{1-s})(1 - q^{2-s}) ... (1 - q^{n-s}).$$

It is again easy to check all parts of the Weil conjectures directly. Complex projective space gives the relevant Betti numbers, which nearly determine the answer.

The number of points on the projective line and projective space are so easy to calculate because they can be written as disjoint unions of a finite number of copies of affine spaces. It is also easy to prove the Weil conjectures for other spaces, such as Grassmannians and flag varieties, which have the same 'paving' property.

9.6. Elliptic curves.

These give the first non-trivial cases of the Weil conjectures (proved by Hasse). If E is an elliptic curve over a finite field with q elements, then the number of points of E defined over the field with q^m elements is $1 - \alpha^m - \beta^m + q^m$, where α and β are complex conjugates with absolute value $\sqrt{q}$. The zeta function is

$$\zeta(E, s) = (1 - \alpha q^{-s})(1 - \beta q^{-s})/(1 - q^{-s})(1 - q^{1-s}).$$

9.7. Weil cohomology.

Weil suggested that the conjectures would follow from the existence of a suitable 'Weil cohomology' for varieties over finite fields, similar to the usual cohomology with rational coefficients for complex varieties. His idea was that if F is the Frobenius automorphism over the finite field, then the number of points of the variety X over the field of order q^m is the finite field, then the number of points of the variety X over the field of order q^m is the number of fixed points of F^m acting on all points of the variety X defined over the algebraic closure. In algebraic topology the number of fixed points of an automorphism can be worked out using the Lefschetz fixed point theorem, given as an alternating sum of traces on the cohomology groups. So if there were similar cohomology groups for varieties over finite fields, then the zeta function could be expressed in terms of them.

The first problem with this is that the coefficient field for a Weil cohomology theory cannot be the rational numbers. To see this consider the case of a supersingular elliptic curve over a finite field of characteristic p. The endomorphism ring of this is an order in a quaternion algebra over the rationals, and should act on the first cohomology group, which should be a 2-dimensional vector space over the coefficient field by analogy with the case of a complex elliptic curve. However a quaternion algebra over the rationals cannot act on a 2-dimensional vector space over the rationals. The same argument eliminates the possibility of the coefficient field being the reals or the p-adic numbers, because the quaternion algebra is still a division algebra over these fields. However it does not eliminate the possibility that the coefficient field is the field of l-adic numbers for some prime $l \neq p$, because over these fields the division algebra splits and becomes a matrix algebra, which can act on a 2-dimensional vector space. Grothendieck and Michael Artin managed to construct suitable cohomology theories over the field of l-adic numbers for each prime $l \neq p$, called l-adic cohomology.

9.8. Grothendieck's proofs of three of the four conjectures.

By the end of 1964 Grothendieck together with Artin and Jean-Louis Verdier proved the Weil conjectures apart from the most difficult third conjecture above, the 'Riemann hypothesis' conjecture (Grothendieck 1965). The general theorems about étale cohomology allowed Grothendieck to prove an analogue of the Lefschetz fixed point formula for the l-adic cohomology theory, and by applying it to the Frobenius automorphism F he was able to prove the conjectured formula for the zeta function:

$$\zeta(s) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

where each polynomial P_i is the determinant of $I - TF$ on the l-adic cohomology group H^i.

The rationality of the zeta function follows immediately. The functional equation for the zeta function follows from Poincaré duality for l-adic cohomology, and the relation with complex Betti numbers of a lift follows from a comparison theorem between l-adic and ordinary cohomology for complex varieties.

More generally, Grothendieck proved a similar formula for the zeta function or 'generalised L-function' of a sheaf F_0:

$$Z(X_0, F_0, t) = \prod_{Z \in X_0} \det\left(1 - F_x^* t^{\deg(x)} \mid F_0\right)^{-1}$$

as a product over cohomology groups:

$$Z(X_0, F_0, t) = \prod_i \det\left(1 - F^* | H_c^i(F)\right)^{(-1)^{i+1}}.$$

The special case of the constant sheaf gives the usual zeta function.

9.9. Deligne's first proof of the Riemann hypothesis conjecture.

Verdier (1974), Serre (1975), Katz (1976) and Freitag & Kiehl (1988) gave expository accounts of the first proof of Deligne (1974). Much of the background in l-adic cohomology is described in (Deligne 1977).

Deligne's first proof of the remaining third Weil conjecture (the 'Riemann hypothesis conjecture') used the following steps:

Use of Lefschetz pencils.

Grothendieck expressed the zeta function in terms of the trace of Frobenius on l-adic cohomology groups, so the Weil conjectures for a d-dimensional variety V over a finite field with q elements depend on showing that the eigenvalues α of Frobenius acting on the ith l-adic cohomology group $H^i(V)$ of V have absolute values $|\alpha|=q^{i/2}$ for an embedding of the algebraic elements of $\mathbf{Q}_l$ into the complex numbers.

After blowing up V and extending the base field, one may assume that the variety V has a morphism onto the projective line $\mathbf{P}^1$, with a finite number of singular fibers with very mild (quadratic) singularities. The theory of monodromy of Lefschetz pencils, introduced for complex varieties and ordinary cohomology by Lefschtz (1974), and extended by Grothendieck (1974) and Deligne & Katz (1973) to l-adic cohomology, relates the cohomology of V to that of its fibers. The relation depends on the space E_x of *vanishing cycles*, the subspace of the cohomology $H^{d-1}(V_x)$ of a non-singular fiber V_x, spanned by classes that vanish on singular fibers.

The Leray spectral sequence relates the middle cohomology group of V to the cohomology of the fiber and base. The hard part to deal with is more or less a group $H^1(P^1, j_*E) = H_c^1(U, E)$, where U is the points the projective line with non-singular fibers, and j is the inclusion of U into the projective line, and E is the sheaf with fibers the spaces E_x of vanishing cycles.

The key estimate.

The heart of Deligne's proof is to show that the sheaf E over U is pure, in other words to find the absolute values of the eigenvalues of Frobenius on its stalks. This is done by studying the zeta functions of the even powers E^k of E and applying Grothendieck's formula for the zeta functions as alternating products over cohomology groups. The crucial idea of considering even k powers of E was inspired by the paper Rankin (1939), who used a similar idea with k = 2 for bounding the Ramanujan tau function. Langlands (1970, section 8) pointed out that a generalisation of Rankin's result for higher even values of k would imply the Ramanujan conjecture, and Deligne realised that in the case of zeta functions of varieties, Grothendieck's theory of zeta functions of sheaves provided an analogue of this generalisation.

The poles of the zeta function of E^k are found using Grothendieck's formula

$$Z(U, E^k, T) = \frac{\det\left(1 - F^*|H_c^1(E^k)\right)}{\det\left(1 - F^*|H_c^0(E^K)\right)\det\left(1 - F^*|H_c^2(E^K)\right)}$$

and calculating the cohomology groups in the denominator explicitly. The H_c^0 term is usually just 1 as U is usually not compact, and the H_c^2 can be calculated explicitly as follows. Poincaré duality relates $H_c^2(E^K)$ to $H_c^0(E^K)$, which is in turn the space of covariants of the monodromy group, which is the geometric fundamental group of U acting on the fiber of E^k at a point. The fiber of E has a bilinear form induced by cup product, which is antisymmetric if d is even, and makes E into a symplectic space. This is a little inaccurate: Deligne did later show that $E \cap E^{\perp} = 0$ by using the hard Lefschetz theorem, this requires the Weil conjectures, and the proof of the Weil conjectures really has to use a slightly more complicated argument with $E/E \cap E^{\perp}$ rather than E. An argument of Kazhdan and Margulis shows that the image of the monodromy group acting on E, given by the Picard-Lefschtez formula, is Zariski dense in a symplectic group and therefore has the same invariants, which are well known from classical invariant theory. Keeping track of the action of Frobenius in this calculation shows that its eigenvalues are all $q^{k(d-1)/2+1}$, so the zeta function of $Z(E^k,T)$ has poles only at $T = 1/q^{k(d-1)/2+1}$.

The Euler product for the zeta function of E^k is

$$Z(E^k, T) = \prod_x \frac{1}{2(E_x^k, T)}.$$

If k is *even* then all the coefficients of the factors on the right considered as power series in T are *non-negative*. This follows by writing

$$\det\left(1 - T^{\deg(x)} F_x|E_x\right)^{-1} = \exp\left(\sum_{n>0} \frac{T^n}{n} \mathrm{Trace}(F_x{}^n|E_x)^k\right),$$

and using the fact that the traces of powers of F are rational, so their k powers are non-negative as k is even. Deligne proves the rationality of the traces by relating them to numbers of points of varieties, which are always (rational) integers.

The powers series for $Z(E^k, T)$ converges for T less than the absolute value $1/q^{k(d-1)/2+1}$ of its only possible pole. When k is even the coefficients of all its Euler factors are non-negative, so that each of the Euler factors has coefficients bounded by a constant times the coefficients of $Z(E^k, T)$ and therefore converges on the same region and has no poles in this region. So for k even the polynomials $Z(E^k, x, T)$ have no zeros in this region, or in other words the eigenvalues of Frobenius on the stalks of E^k have absolute value at most $q^{k(d-1)/2+1}$.

This estimate can be used to find the absolute value of any eigenvalue α of Frobenius on a fiber of E as follows. For any integer k, α^k is an eigenvalue of Frobenius on a stalk of E^k, which for k even is bounded by $q^{1+k(d-1)/2}$. So

$$\left|\alpha^k\right| \leq q^{\frac{k(d-1)}{2}+1}.$$

As this is true for arbitrarily large even k, this implies that

$$|\alpha| \leq q^{\frac{(d-1)}{2}}.$$

Poincaré duality then implies that

$$|\alpha| = q^{\frac{(d-1)}{2}}.$$

Completion of the proof.
The deduction of the Riemann hypothesis from this estimate is mostly a fairly straightforward use of standard techniques and is done as follows.

The eigenvalues of Frobenius on $H_c^2(U, E)$ can now be estimated as they are the zeros of the zeta function of the sheaf E. This zeta function can be written as an Euler product of zeta functions of the stalks of E, and using the estimate for the eigenvalues on these stalks shows that this product converges for $|T| < q^{-d/2-1/2}$, so that there are no zeros of the zeta function in this region. This implies that the eigenvalues of Frobenius on E are at most $q^{d/2+1/2}$ in absolute value. In fact it will soon be seen that they have absolute value exactly $q^{d/2}$. This step of the argument is very similar to the usual proof that the Riemann zeta function has no zeros with real part greater than 1, by writing it as an Euler product.

The conclusion of this is that the eigenvalues α of the Frobenius of a variety of even dimension d on the middle cohomology group satisfy

$$|\alpha| \le q^{\frac{(d+1)}{2}}.$$

To obtain the Riemann hypothesis one needs to eliminate the 1/2 from the exponent. This can be done as follows. Applying this estimate to any even power V^k of V and using the Künneth formula shows that the eigenvalues of Frobenius on the middle cohomology of a variety V of any dimension d satisfy

$$|\alpha^k| \le q^{\frac{(kd+1)}{2}}.$$

As this is true for arbitrarily large even k, this implies that

$$|\alpha| \le q^{\frac{d}{2}}.$$

Poincaré duality then implies that

$$|\alpha| = q^{\frac{d}{2}}.$$

This proves the Weil conjectures for the middle cohomology of a variety. The Weil conjectures for the cohomology below the middle dimension follow from this by applying the weak Lefschtetz theorem, and the conjectures for cohomology above the middle dimension then follow from Poincaré duality.

9.10. Deligne's second proof.

Deligne (1980) found and proved a generalization of the Weil conjectures, bounding the weights of the pushforward of a sheaf. In practice it is this generalization rather than the original Weil conjectures that is mostly used in applications, such as the hard Lefschtetz theorem. Much of the second proof is a rearrangement of the ideas of his first proof. The main extra idea needed is an argument closely related to the theorem of Hadamard and de la Vallée Poussin, used by Deligne to show that various L-series do not have zeros with real part 1.

A constructible sheaf on a variety over a finite field is called pure of weight β if for all points x the eigenvalues of the Frobenius at x all have absolute value $N(x)^{\beta/2}$, and is called mixed of weight ≤ β if it can be written as repeated extensions by pure sheaves with weights ≤ β.

Deligne's theorem states that if f is a morphism of schemes of finite type over a finite field, then $R^i f_!$ takes mixed sheaves of weight ≤ β to mixed sheaves of weight ≤ β + i.

The original Weil conjectures follow by taking f to be a morphism from a smooth projective variety to a point and considering the constant sheaf $\mathbf{Q}_l$ on the variety. This gives an upper bound on the absolute values of the eigenvalues of Frobenius, and Poincaré duality then shows that this is also a lower bound.

In general $R^i f_!$ does not take pure sheaves to pure sheaves. However it does when a suitable form of Poincaré duality holds, for example if f is smooth and proper, or if one works with perverse sheaves rather than sheaves as in Beilinson, Bernstein & Deligne (1982).

Inspired by the work of Witten (1982) on Morse theory, Laumon (1987) found another proof, using Deligne's l-adic Fourier transform, which allowed him to simplify Deligne's proof by avoiding the use of the method of Hadamard and de la Vallée Poussin. His proof generalizes the classical calculation of the absolute value of Gauss sums using the fact that the norm of a Fourier transform has a simple relation to the norm of the original function. Kiehl & Weissauer (2001) used Laumon's proof as the basis for their exposition of Deligne's theorem. Katz (2001) gave a further simplification of Laumon's proof, using monodromy in the spirit of Deligne's first proof. Kedlaya (2006) gave another proof using the Fourier transform, replacing étale cohomology with rigid cohomology.