CHAPTER 3

Sphere packings and kissing numbers

3.1. Introduction.

3.2. Sphere packings and kissing numbers.

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The E_8 lattice is remarkable in that it gives optimal solutions to the sphere packing problem and the kissing number problem in 8 dimensions.

The sphere packing problem asks what is the densest way to pack (solid) n-dimensional spheres of a fixed radius in $\mathbf{R}^n$ so that no two spheres overlap. Lattice packings are special types of sphere packings where the spheres are centered at the points of a lattice. Placing spheres of radius $1/\sqrt{2}$ at the points of the E_8 lattice gives a lattice packing in $\mathbf{R}^8$ with a density of

It has long been known that this is the maximum density that can be achieved by a lattice packing in 8 dimensions.[6] Furthermore, the E_8 lattice is the unique lattice (up to isometries and rescalings) with this density.[7] Mathematician Maryna Viazovska has recently shown that this density is, in fact, optimal even among irregular packings.[8][9]

The kissing number problem asks what is the maximum number of spheres of a fixed radius that can touch (or "kiss") a central sphere of the same radius. In the E_8 lattice packing mentioned above any given sphere touches 240 neighboring spheres. This is because there are 240 lattice vectors of minimum nonzero norm (the roots of the E_8 lattice). It was shown in 1979 that this is the maximum possible number in 8 dimensions.[10][11]

The sphere packing problem and the kissing number problem are remarkably difficult and optimal solutions are only known in 1, 2, 3, 8, and 24 dimensions (plus dimension 4 for the kissing number problem). The fact that solutions are known in dimensions 8 and 24 follows in part from the special properties of the E_8 lattice and its 24-dimensional cousin, the Leech lattice.