CHAPTER 5

The adonion $\Pi_{7,1}$ and novanion $\Pi_{9,1}$ Lorentzian lattices.

5.1. Introduction.

5.2. The novanion $\Pi_{9,1}$ Lorentzian lattice.

Except as 13.8 does not seem to be represented in SPLAG.

5.2. The E_8 lattice.

SPLAG 10-, 120-, 340-, 406-, 427-, 461-.

For any basis of $\mathbb{R}^n$, the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice, which can be viewed as a regular tiling of a space by a primitive cell called the fundamental parallelotope. A *unimodular* lattice is an integral lattice with determinant or hypervolume 1 or −1.

The E_8 lattice is a special lattice in $\mathbb{R}^8$. It can be characterised as the unique positive-definite, even unimodular lattice of rank 8, so that it can be generated by the columns of an 8×8 matrix where the determinant of the fundamental parallelotope of the lattice is ±1. The name derives from the fact that it is the root lattice of the E_8 root system, described in [Ad15], chapter IV.

The *norm* of the E_8 lattice (divided by 2) is a positive definite even unimodular quadratic form in 8 variables, and conversely such a quadratic form can be used to construct a positive-definite, even, unimodular lattice of rank 8. The existence of such a form was first shown by H. Smith in 1867 and the first explicit construction of this quadratic form was given by A. Korkin and G. Zoltarev in 1873. The E_8 lattice is also called the *Gosset lattice* after T. Gosset who was one of the first to study the geometry of the lattice around 1900.

The E_8 lattice is a discrete subgroup of $\mathbb{R}^8$ which spans all of $\mathbb{R}^8$. It can be given explicitly by the set of points $\Gamma_8 \subset \mathbb{R}^8$ such that

- all the coordinates are integers or half-integers but not a mixture of these
- the sum of the eight coordinates is an even integer.

The sum of two lattice points is another lattice point, so that Γ_8 is indeed a subgroup.

An alternative description of the E_8 lattice is the set of all points in $\Gamma'_8 \subset \mathbb{R}^8$ such that

- all the coordinates are integers and the sum of the coordinates is even, or
- all the coordinates are half-integers and the sum of the coordinates is odd.

The lattices Γ_8 and Γ'_8 are isomorphic – we can pass from one to the other by changing the signs of any odd number of coordinates. The lattice Γ_8 is called the *even coordinate system* for E_8 while the lattice Γ_8' is called the *odd coordinate system*. Unless we specify otherwise we will work in the even coordinate system.

Even unimodular lattices occur only in dimensions divisible by 8. In dimension 16 there are two such lattices: $\Gamma_8 \oplus \Gamma_8$, and Γ_{16} constructed analogously to Γ_8. In dimension 24 there are 24 such lattices, called Niemeier lattices, the most important of which is the Leech lattice.

A possible basis for Γ_8 is given by the columns of the upper triangular matrix

$$\Gamma_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Γ_8 is then the integral span of these vectors. All other possible bases are obtained from this one by right multiplication by elements of GL(8, $\mathbb{Z}$).

The shortest nonzero vectors in Γ_8 have norm 2. There are 240 such vectors.

- All half-integer: (can only be ±1/2)
  - All positive or all negative: 2
  - Four positive, four negative: (8×7×6×5)/(4×3×2×1) = 70
  - Two of one, six of the other: 2× (8×7)/(2×1) = 56
- All integer: (can only be 0, ±1)
  - Two ±1, six zeroes: 4×(8×7)/(2×1) = 112

These form a root system of type E_8. The lattice Γ_8 is equal to the E_8 root lattice, which means that it is given by the integral span of the 240 roots. Any choice of 8 simple roots gives a basis for Γ_8.

Subsequent text will be revised substantially

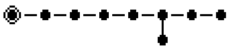
The automorphism group (or symmetry group) of a lattice in $\mathbb{R}^n$ is defined as the subgroup of the orthogonal group O(n) that preserves the lattice. The symmetry group of the E_8 lattice is the Weyl/Coxeter group of type E_8. This is the group generated by reflections in the hyperplanes orthogonal to the 240 roots of the lattice. Its order is given by

The E_8 Weyl group contains a subgroup of order 128·8! consisting of all permutations of the coordinates and all even sign changes. This subgroup is the Weyl group of type D_8. The full E_8 Weyl group is generated by this subgroup and the block diagonal matrix $H_4 \oplus H_4$ where H_4 is the Hadamard matrix

Geometry[edit]

See 5_{21} honeycomb

The E_8 lattice points are the vertices of the 5_{21} honeycomb, which is composed of regular 8-simplex and 8-orthoplex facets. This honeycomb was first studied by Gosset who called it a *9-ic semi-regular figure*[4] (Gosset regarded honeycombs in n dimensions as degenerate n+1 polytopes). In Coxeter's notation,[5] Gosset's honeycomb is denoted by 5_{21} and has the Coxeter-Dynkin diagram:



This honeycomb is highly regular in the sense that its symmetry group (the affine Weyl group) acts transitively on the *k*-faces for $k \leq 6$. All of the *k*-faces for $k \leq 7$ are simplices.

The vertex figure of Gosset's honeycomb is the semiregular E_8 polytope (4_{21} in Coxeter's notation) given by the convex hull of the 240 roots of the E_8 lattice.

Each point of the E_8 lattice is surrounded by 2160 8-orthoplexes and 17280 8-simplices. The 2160 deep holes near the origin are exactly the halves of the norm 4 lattice points. The 17520 norm 8 lattice points fall into two classes (two orbits under the action of the E_8 automorphism group): 240 are twice the norm 2 lattice points while 17280 are 3 times the shallow holes surrounding the origin.

A hole in a lattice is a point in the ambient Euclidean space whose distance to the nearest lattice point is a local maximum. (In a lattice defined as a uniform honeycomb these points correspond to the centers of the facets volumes.) A deep hole is one whose distance to the lattice is a global maximum. There are two types of holes in the E_8 lattice:

- *Deep holes* such as the point (1,0,0,0,0,0,0,0) are at a distance of 1 from the nearest lattice points. There are 16 lattice points at this distance which form the vertices of an 8-orthoplex centered at the hole (the Delaunay cell of the hole).

- *Shallow holes* such as the point are at a distance of from the nearest lattice points. There are 9 lattice points at this distance forming the vertices of an 8-simplex centered at the hole.

Applications[edit]

In 1982 Michael Freedman produced a bizarre example of a topological 4-manifold, called the E_8 manifold, whose intersection form is given by the E_8 lattice. This manifold is an example of a topological manifold which admits no smooth structure and is not even triangulable.

In string theory, the heterotic string is a peculiar hybrid of a 26-dimensional bosonic string and a 10-dimensional superstring. In order for the theory to work correctly, the 16 mismatched dimensions must be compactified on an even, unimodular lattice of rank 16. There are two such lattices: $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} (constructed in a fashion analogous to that of Γ_8). These lead to two version of the heterotic string known as the $E_8 \times E_8$ heterotic string and the SO(32) heterotic string.

5.3. The E_8 and novanion group constructions from octonions.

[To be modified.]

The E_8 lattice is closely related to the nonassociative algebra of real octonions **O**. It is possible to define the concept of an integral octonion analogous to that of an integral quaternion. The integral octonions naturally form a lattice inside **O**. This lattice is just a rescaled E_8 lattice. (The minimum norm in the integral octonion lattice is 1 rather than 2). Embedded in the octonions in this manner the E_8 lattice takes on the structure of a nonassociative ring.

Fixing a basis (1, *i*, *j*, *k*, ℓ, ℓ*i*, ℓ*j*, ℓ*k*) of unit octonions, one can define the integral octonions as a maximal order containing this basis. (One must, of course, extend the definitions of *order* and *ring* to include the nonassociative case). This amounts to finding the largest subring of **O** containing the units on which the expressions x^*x (the norm of *x*) and $x + x^*$ (twice the real part of *x*) are integer-valued. There are actually seven such maximal orders, one corresponding to each of the seven imaginary units. However, all seven maximal orders are isomorphic. One such maximal order is generated by the octonions *i*, *j*, and ½ ($i + j + k + \ell$).

A detailed account of the integral octonions and their relation to the E_8 lattice can be found in Conway and Smith (2003).

Example definition of integral octonions[edit]

Consider octonion multiplication defined by triads: 137, 267, 457, 125, 243, 416, 356. Then integral octonions form vectors:

1) , i=0, 1, ..., 7

2) , indexes abc run through the seven triads 124, 235, 346, 457, 561, 672, 713

3) , indexes pqrs run through the seven tetrads 3567, 1467, 1257, 1236, 2347, 1345, 2456.

Imaginary octonions in this set, namely 14 from 1) and 7*16=112 from 3), form the roots of the Lie algebra . Along with the remaining 2+112 vectors we obtain 240 vectors that form roots of Lie algebra . See the Koca work on this subject.[13]

If we look at octonions, they have a multiplicative identity 1, and seven units $e_1, e_2, \dots, e_7$. If we ignore their additive structure for the moment, we also have plus or minus these elements, and the plus and minus are distinguishable in the multiplicative group structure. The total number of elements of these types is now 16.

If we go over to a ring structure, additively it contains zero. We have also seen that the octonions with Lie brackets given by [A, B] = AB – BA, where for example this can be a matrix representation, although octonions are not matrices, since they are not associative, nevertheless they can be given the above Lie algebra structure (mod 12), satisfying the Jacobi identities.

The elements contain zero, for example $6e_7 \times 2e_7 = 0$ (mod 12). We therefore need to create a multiplicative group without zero. Note we can proceed not only (mod 12), but as we have mentioned, (mod 6), (mod 4), (mod 3) and (mod 2). If we take the case (mod 6), since for example if $3 \times 2 = 0$, then $3^{-1} \times 3 \times 2 = 0$, so $2 = 0$ and has no inverse. The only occurrence of multiplicative terms $\neq 0$ for an abelian group with result zero is 3×2 (mod 6). Then if we specify a modification of the group so that $3 \times 2 = 1$, since by prime factorisation this is the only pair $\neq 0$ that gives zero, we can define this new allocation in which $3^{-1} = 2$ and $2^{-1} = 3$, and this retains the structure of the Lie algebra, and the multiplicative group defined in this way does not contain zero, which we have excluded from it. A more general way of stating this is that we replace 0 (mod 6) in the Lie algebra by ± 1 in its group.

To consider (mod 12), the products which are zero (mod 12) contain powers of factors 3 and 2, where 3 and 3^2 occur (mod 12) and so do 2, 2^2 and 2^3. To analyse this, first look at (mod 9) where the only product of terms $\neq 0$ that has a product 0 (mod 9) is 3×3. So a Lie algebra (mod 9) can give rise to a multiplicative group where $3 \times 3 = 1$, in which 3 is its own inverse. Likewise for (mod 8) = (mod 2^3), the only terms $\neq 0$ with abelian product zero are 4×2. The element 4 is now set with inverse 2, and on replacing zero in the (mod 8) Lie algebra by ± 1 as an element in its group, we again obtain a multiplicative group. For (mod 12) we consider elements (mod 9×8) = (mod 72). Then there exists a corresponding group in which elements are classified multiplicatively by their (mod 9) or (mod 8) counterparts where 9 is irreducible in terms of 8. We could choose for example an element (mod 72) as $3^m \times 2^n \times$(prime < 72) where 3^m is in (mod 9), 2^n is in (mod 8) and the prime is considered (mod 72). Then from the Lie algebra (mod 12) derived from (mod 72), on setting zero to ± 1 the mapping to the group is obtained.

If 9 is reducible in terms of 8, then consider again the case where 1 is identified with zero (mod 12), so that the multiplicative algebra at this stage contains 11 elements. Then $3 \times 4 = 1$, so $3 = 4^{-1} = 2^{-2}$. We have $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $5 = 2^4$, $10 = 2^5$, $9 = 2^6$, $7 = 2^7$, $3 = 2^8$, $6 = 2^9$ and $1 = 2^{10}$. To check consistency, for example $2^6 = 9 = 3 \times 3 = 2^8 \times 2^8 = 2^{16} = 2^{10} \times 2^6 = 2^6$. Then 2 is represented by the cyclic permutation (1 2 4 8 5 10 9 7 3 6), and the reduction of 3 in terms of the generator 2 gives a multiplicative group on 10 elements. If we were to choose $3 \times 4 = -1$, then $3 = -2^{-2} = 2^8$, so $2^{10} = -1$, which is the mapping $2 \rightarrow 2i$ from the previous case.

Consider the original elements $e_7, e_6, \ldots, 1, -1, -e_1, \ldots, -e_7$, 16 elements in all, and zero. If we allow new elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$, with $12e_r = 0e_r$, so that this exists in a (mod 12) arithmetic, then the Lie algebra derived from the octonions in this way has first a set of 11 elements to choose from (that is, not 0). Then if we create new elements by adding a second set, because the addition is $ae_r + be_s$ with $e_r \neq e_s$, but $ae_r + a(-e_r) = 0$, the paired e_r and $(-e_r)$ elements (mod 12) have 12 values (including $0e_r + 0(-e_r) = 0$) that we must exclude, but $-[0e_r + 0(-e_r)] = 0e_r + 0(-e_r) = 0$ is the same case. Thus we have 12 – 6 elements that are not zero. Further, we have values

(1) $1e_7 + 0(-e_7) = 1e_7$

(2) $2e_7 + 1(-e_7) = 1e^7$

... ...

(11) $11e_7 + 10(-e_7) = 1e_7$

(12) $12e_7$ [which is $0(e_7)$] $+ 11(-e_7) = 1e_7$,

which we must also collect together as one item, but case (1) (mod 12) is the same as case (12), etc, with case (6) the same as case (7). Thus there are 12/2 = 6 distinct cases for $1e_7$ and the total number of exclusions for differences $2e_7$, $3e_7$, ..., $11e_7$ is the same. Thus for the pair e_7, ($-e_7$) we obtain $12^2 - 12\times6 = 12\times6$ possibilities. There are 8 values e_7, e_6, ... e_1, 1 that we can account for in a similar way. Then the total number of distinct possibilities is $(12\times6)^8$, but we must exclude the cases where we have a sum of all the e_r equal to zero. For e_7 this is 12 cases, as we have seen, and likewise for e_6, ... e_1, 1. Thus we exclude $(12)^8$ items, and the total number of elements of the octonion algebra is

$$(12\times6)^8 - 12^8 = 12^8(6^8 - 1).$$

This is the order of the Lie algebra we have derived from the octonions. We obtain the order of the Lie group from it. ☐

5.4. Novanion simple group constructions. [To be modified.]

Our objective now is to convert novanion algebras, with novanion brackets (mod m) like a Lie bracket, to novanion algebras, and then map these novanion algebras to groups. These groups, being multiplicatively invertible, do not contain zero, so we map $0 \rightarrow \pm1$, which retains the novanion algebra structure. The resulting novanion group thus does not have the same structure as the multiplicative part of the ring mapped directly to the novanion algebra.

For the 10-novanions consider the elements e_9, e_8, ... , 1, -1, $-e_1$, ... , $-e_9$, 20 elements in all, and zero. Using elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$ again, then in the case when we are not using ±1, the extended Lie group derived from the 10-novanions has in a calculation similar to one given for the octonions, $(12\times6)^9$ elements. If we assume the scalar part is not zero, the available scalar values are ±1, ±2, ..., ±11, amounting to 22 components, but we have $1 = -11$ (mod 12), etc., so effectively there are 11 components. We know that when the scalar part is not zero, we cannot get a zero result on multiplying 10-novanions together, provided we ignore the (mod 12) restriction. Thus the order of the 10-novanion group derived from this algebra is $(12\times6)^9\times11$ elements. If we obtain 0 (mod 12) in any multiplication, then as before we will map this result to ±1 without changing the Lie-type bracket algebra.

For the 26-novanions or indeed any n-novanion algebra, the case is similar with $(12\times6)^n\times11$ elements. We know that overrides derived from the octonion structure operate for all 10-novanion components in a 26-novanion, thus external to the 10-novanion or an octonion or their subalgebras, there are no other proper subentities in this algebra, which implies that the groups defined by a quotient of a 10-novanion group or an octonion group with the 26-novanion group are simple.

Note that we have demonstrated in section 12 that both bosonic and fermionic structures can be implemented in novanions. Thus any argument which says that a bosonic 26-novanion cannot contain a 10-novanion is incorrect. Since the bosonic 10-novanion is a maximal proper subgroup of the 26-novanions, the 26-novanions do not generate a simple group directly, only via quotients. ☐

Note that $26 - 10 = 16$, but we have no indication that this is two copies of E_8. Indeed, it cannot be, since the structure is not closed, encroaching in overrides onto the 10-novanion. So we show once again that the novanion groups do not lead to the compactification assumed in string theory of the identification of the 16 dimensions with $E_8 \times E_8$. ☐

These considerations may be extended without limit, for example to the 80-novanions, or more generally the $(3^n - 1)$-novanions, $n > 0$, which are those hypercube novanions with one corner containing not a triplet, but a singlet. In particular, for the 80-novanions there exists a 28-

novanion slice group which has no containing groups other than the 80-novanion group itself, so there is an analogous structure to that for the 26-novanion group. The remaining dimensions are $79 - 27 = 52$, and this generates a simple group via the quotient with the 28-novanion group. ☐