CHAPTER 7

The novanion and adonion $\Pi_{25,1}$ Lorentzian lattices.

7.1. Introduction.

7.2. The novanion and adonion $\Pi_{25,1}$ Lorentzian lattices.

The Leech lattice can also be constructed as where *w* is the Weyl vector:

in the 26-dimensional even Lorentzian unimodular lattice $II_{25,1}$.

By Faulhaber's formula

$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

The existence of such an integral vector of norm zero relies on the fact that $1^2 + 2^2 + ... + 24^2$ is a perfect square (in fact 70^2); the number 24 is the only integer bigger than 1 with this property. This was conjectured by Édouard Lucas, but the proof came much later, based on elliptic functions.

The vector in this construction is really the Weyl vector of the even sublattice D_{24} of the odd unimodular lattice I^{25}. More generally, if L is any positive definite unimodular lattice of dimension 25 with at least 4 vectors of norm 1, then the Weyl vector of its norm 2 roots has integral length, and there is a similar construction of the Leech lattice using L and this Weyl vector.

7.3. The Leech lattice.

The *Leech lattice* is an even unimodular lattice Λ_{24} in 24-dimensional Euclidean space, which is one of the best models for the kissing number problem. It was discovered by John Leech (1967). It may also have been discovered (but unpublished) by Ernst Witt in 1940.

Characterization[edit]

The Leech lattice Λ_{24} is the unique lattice in $\mathbf{E}^{24}$ with the following list of properties:

- It is unimodular; i.e., it can be generated by the columns of a certain 24×24 matrix with determinant 1.
- It is even; i.e., the square of the length of each vector in Λ_{24} is an even integer.
- The length of every non-zero vector in Λ_{24} is at least 2.

The last condition is equivalent to the condition that unit balls centered at the points of Λ_{24} do not overlap. Each is tangent to 196,560 neighbors, and this is known to be the largest number of non-overlapping 24-dimensional unit balls that can simultaneously touch a single unit ball (compare with 6 in dimension 2, as the maximum number of pennies which can touch a central penny; see kissing number). This arrangement of 196560 unit balls centred about another unit ball is so efficient that there is no room to move any of the balls; this configuration, together with its mirror-image, is the *only* 24-dimensional arrangement where 196560 unit balls simultaneously touch another. This property is also true in 1, 2 and 8 dimensions, with 2, 6 and 240 unit balls, respectively, based on the integer lattice, hexagonal tiling and E8 lattice, respectively.

It has no root system and in fact is the first unimodular lattice with no *roots* (vectors of norm less than 4), and therefore has a centre density of 1. By multiplying this value by the volume

of a unit ball in 24 dimensions, , one can derive its absolute density.

Conway (1983) showed that the Leech lattice is isometric to the set of simple roots (or the Dynkin diagram) of the reflection group of the 26-dimensional even Lorentzian unimodular lattice $II_{25,1}$. By comparison, the Dynkin diagrams of $II_{9,1}$ and $II_{17,1}$ are finite.

Constructions[edit]

The Leech lattice can be constructed in a variety of ways. As with all lattices, it can be constructed by taking the integral span of the columns of its generator matrix, a 24×24 matrix with determinant 1.

Leech generator matrix [show]

[1]

Using the binary Golay code[edit]

The Leech lattice can be explicitly constructed as the set of vectors of the form $2^{-3/2}(a_1, a_2, ..., a_{24})$ where the a_i are integers such that

and for each fixed residue class modulo 4, the 24 bit word, whose 1s correspond to the coordinates i such that a_i belongs to this residue class, is a word in the binary Golay code. The Golay code, together with the related Witt design, features in a construction for the 196560 minimal vectors in the Leech lattice.

Based on other lattices[edit]

Conway & Sloane (1982) described another 23 constructions for the Leech lattice, each based on a Niemeier lattice. It can also be constructed by using three copies of the E8 lattice, in the same way that the binary Golay code can be constructed using three copies of the extended Hamming code, H_8. This construction is known as the *Turyn* construction of the Leech lattice.

As a laminated lattice[edit]

Starting with a single point, Λ_0, one can stack copies of the lattice Λ_n to form an (*n* + 1)-dimensional lattice, Λ_{n+1}, without reducing the minimal distance between points. Λ_1 corresponds to the integer lattice, Λ_2 is to the hexagonal lattice, and Λ_3 is the face-centered cubic packing. Conway & Sloane (1982b) showed that the Leech lattice is the unique laminated lattice in 24 dimensions.

As a complex lattice[edit]

The Leech lattice is also a 12-dimensional lattice over the Eisenstein integers. This is known as the *complex Leech lattice*, and is isomorphic to the 24-dimensional real Leech lattice. In the complex construction of the Leech lattice, the binary Golay code is replaced with the ternary Golay code, and the Mathieu group M_{24} is replaced with the Mathieu group M_{12}. The E_6 lattice, E_8 lattice and Coxeter–Todd lattice also have constructions as complex lattices, over either the Eisenstein or Gaussian integers.

Using the icosian ring[edit]

The Leech lattice can also be constructed using the ring of icosians. The icosian ring is abstractly isomorphic to the E8 lattice, three copies of which can be used to construct the Leech lattice using the Turyn construction.

Witt's construction[edit]

In 1972 Witt gave the following construction, which he said he found in 1940 January 28.

Suppose that *H* is an *n* by *n* Hadamard matrix, where *n*=4*ab*. Then the matrix defines a bilinear form in 2*n* dimensions, whose kernel has *n* dimensions. The quotient by this kernel is a nonsinguar bilinear form taking values in (1/2)**Z**. It has 3 sublattices of index 2 that are integral bilinear forms. Witt obtained the Leech lattice as one of these three sublattices by taking *a*=2, *b*=3, and taking *H* to be the 24 by 24 matrix (indexed by **Z**/23**Z** ∪ ∞) with entries X(*m*+*n*) where X(∞)=1, X(0)=−1, X(*n*)=is the quadratic residue symbol mod 23 for nonzero *n*. This matrix *H* is a Paley matrix with some insignificant sign changes.

Using a Paley matrix[edit]

Chapman (2001) described a construction using a skew Hadamard matrix of Paley type. The

Niemeier lattice with root system can be made into a module for the ring of integers of

the field . Multiplying this Niemeier lattice by a non-principal ideal of the ring of integers gives the Leech lattice.

Using octonions[edit]

If *L* is the set of octonions with coordinates on the lattice. Then the Leech lattice is the

set of triplets such that:

where

Symmetries[edit]

The Leech lattice is highly symmetrical. Its automorphism group is the Conway group Co_0, which is of order 8 315 553 613 086 720 000. The center of Co_0 has two elements, and the quotient of Co_0 by this center is the Conway group Co_1, a finite simple group. Many other sporadic groups, such as the remaining Conway groups and Mathieu groups, can be constructed as the stabilizers of various configurations of vectors in the Leech lattice.

Despite having such a high *rotational* symmetry group, the Leech lattice does not possess any hyperplanes of reflection symmetry. In other words, the Leech lattice is chiral.

The automorphism group was first described by John Conway. The 398034000 vectors of norm 8 fall into 8292375 'crosses' of 48 vectors. Each cross contains 24 mutually orthogonal vectors and their negatives, and thus describe the vertices of a 24-dimensional orthoplex. Each of these crosses can be taken to be the coordinate system of the lattice, and has the same symmetry of the Golay code, namely $2^{12} \times |M_{24}|$. Hence the full automorphism group of the Leech lattice has order 8292375 × 4096 × 244823040, or 8 315 553 613 086 720 000.

Geometry[edit]

Conway, Parker & Sloane (1982) showed that the covering radius of the Leech lattice is ; in other words, if we put a closed ball of this radius around each lattice point, then these just cover Euclidean space. The points at distance at least from all lattice points are called the ***deep holes*** of the Leech lattice. There are 23 orbits of them under the automorphism group of the Leech lattice, and these orbits correspond to the 23 Niemeier lattices other than the Leech lattice: the set of vertices of deep hole is isometric to the affine Dynkin diagram of the corresponding Niemeier lattice.

The Leech lattice has a density of . Cohn & Kumar (2009) showed that it gives the densest lattice packing of balls in 24-dimensional space. Henry Cohn, Abhinav Kumar, and Stephen D. Miller et al. (2016) improved this by showing that it is the densest sphere packing, even among non-lattice packings.

The 196560 minimal vectors are of three different varieties, known as *shapes*:

- 1104 vectors of shape $(4^2,0^{22})$, for all permutations and sign choices;
- 97152 vectors of shape $(2^8,0^{16})$, where the '2's correspond to octads in the Golay code, and there is an even number of minus signs;
- 98304 vectors of shape $(-3,1^{23})$, where the changes of signs come from the Golay code, and the '3' can appear in any position.

The ternary Golay code, binary Golay code and Leech lattice give very efficient 24-dimensional spherical codes of 729, 4096 and 196560 points, respectively. Spherical codes are higher-dimensional analogues of Tammes problem, which arose as an attempt to explain the distribution of pores on pollen grains. These are distributed as to maximise the minimal angle between them. In two dimensions, the problem is trivial, but in three dimensions and higher it is not. An example of a spherical code in three dimensions is the set of the 12

vertices of the regular icosahedron.

History[edit]

Many of the cross-sections of the Leech lattice, including the Coxeter–Todd lattice and Barnes–Wall lattice, in 12 and 16 dimensions, were found much earlier than the Leech lattice. O'Connor & Pall (1944) discovered a related odd unimodular lattice in 24 dimensions, now called the odd Leech lattice, one of whose two even neighbors is the Leech lattice. The Leech lattice was discovered in 1965 by John Leech (1967, 2.31, p. 262), by improving some earlier sphere packings he found (Leech 1964).

Conway (1968) calculated the order of the automorphism group of the Leech lattice, and, working with John G. Thompson, discovered three new sporadic groups as a by-product: the Conway groups, Co_1, Co_2, Co_3. They also showed that four other (then) recently announced sporadic groups, namely, Higman-Sims, Suzuki, McLaughlin, and the Janko group J_2 could be found inside the Conway groups using the geometry of the Leech lattice. (Ronan, p. 155)

> Bei dem Versuch, eine Form aus einer solchen Klasse wirklich anzugeben, fand ich mehr als 10 verschiedene Klassen in Γ_{24}
>
> *Witt (1941, p. 324)*

Witt (1941, p. 324), has a single rather cryptic sentence mentioning that he found more than 10 even unimodular lattices in 24 dimensions without giving further details. Witt (1998, p. 328–329) stated that he found 9 of these lattices earlier in 1938, and found two more, the Niemeier lattice with A_1^{24} root system and the Leech lattice (and also the odd Leech lattice), in 1940.