Electroweak chirality, 10-novanions and the heterotic string

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1. Introduction.

We introduce novanion physics, a mathematical part of which contains embedded within it the algebra of quaternions. We interpret these quaternions as fixed with a base point which at time t = 0 and space points xi, yj and zk = 0 is located at the beginning of the universe. However, unlike the quaternions, the novanions in general are nonassociative. This algebra is new, develops mathematical understanding at the present time of writing, and we are able to prove its correctness. These novanion algebras have the property that two such novanions multiplied together cannot become zero unless one of them is zero, provided and only provided that $t \neq 0$.

Our interpretation is that at time t, xi, yi and zj = 0 alone, division by zero becomes possible for the quaternion part of this physics, and this corresponds to non-conservation of number at the creation of the universe, under allowable multiplicative interaction of quaternionic quantum states.

For the encompassing novanionic part, at time t = 0, at novanionic space components at this time, the algebra has a rich structure in which zero becomes the product of novanion pairs. This is interpreted as the creation of the universe from nothing and has a spacelike extension at t = 0, unlike the quaternions.

An interesting and essential feature is that at $t \neq 0$ the novanions become division algebras, that is, nonzero entities cannot be reduced by multiplicative interaction back to a zero state.

Further, states of negative and positive time are present in the theory. For positive time, we investigate the interaction given by multiplication between positive time states, and for negative time the corresponding negative theory. It is the case that t^2 is positive, and so is $(-t)^2$ so that if the space components are small in comparison with t, the multiplicative interaction for the negative time component results in an appearance in the positive time universe. The other interactions in the negative time universe we consider for the moment to be addition, which retains its negative t component. We will interpret such features as additive interaction being quantum mechanical, and multiplicative interaction corresponding, in a terminology we would prefer to reject, to the collapse of the wave function.

Quaternions have a cyclic algebra on their basis vectors i, j and k, and any distinct combination of two of these anticommutes. We investigate the algebra of the exquaternions. For the exquaternions, the algebra is not entirely cyclic, one arrow is reversed. The result is that the exquaternions are nonassociative, and we are able to prove by an example that they do not form a division algebra. We will say that interaction given by multiplication of two states described by exquaternions can result in the zero state, and these states, present in the early universe, have self-annihilated. Allocating a quaternion, in the method to be described, to a lepton, results in the conclusion that these particles have handedness, and this chirality is observed experimentally. Only left-handed fermions participate in the weak interaction, while there are no right-handed neutrinos.

Novanions have a norm, in the case of quaternions given by

$$s^2 = c^2t^2 - (xi)^2 - (yj)^2 - (zk)^2,$$

and we interpret this norm in terms of relativity, in which time is real and space states are quaternionic imaginary. In the general case the norm relates to non-real anticommuting parts except for self-squares of -1, so that apart from the number of dimensions the norm given by multiplicative interaction is an extension of the above type. We use this also in a relativistic extension of the wave function, and the correctness or incorrectness of this allocation should be determinable experimentally.

We interpret the ten dimensional 10-novanions, where $10 = 1 + 3^2$, as a triplet of quaternions, each triplet of which is defined in an interaction algebra with another triplet. Here our interpretation is that the triplet corresponds to the three families of leptons, and that further, these are not pure states, so that neutrinos can mix, as is verified experimentally.

Mathematically, the n-novanions were allocated initially an algebra of dimension $1 + 3^f7^h$, where the 7 component describes the octonions, given by the nonassociative Fano plane.

We introduce the idea that 10-novanions correspond to the 10-dimensional heterotic string, in other words, there is a novanionic relationship to anomaly cancellation and the existence of finite quantum states.

It then became an interesting question of whether the 26-dimensional bosonic string can be represented by an algebra of 26-novanion type. For this to be feasible, we have first tried the number 26 as $1 + 5^2$, so this reduces to the study of an algebra on pentagonal diagrams. By a theorem, all associative division algebras have dimension ≤ 4, so these are nonassociative. The result of these computations is that there are no such pentagonal diagrams, allocated to bosons, forming a division algebra. However, we are able to persist, and we allocate 26 as the number 1 (time) + 7 (from the space components of the octonions) + 9 + 9, where the number of space components of the 10-novanions is 9. This yields a novanion algebra.

2. States and transformations.

We sketch two ideas from John Bell's *Speakable and unspeakable in quantum mechanics*.

My philosophy is that states are the underlying structure of a physical system, which is determinate, and observations are embedded within this physical system as transformations of states. Observations are then the consequence of a physical theory, and not its basis.

Bell describes the de Broglie Bohm interpretation of quantum mechanics, and its non-conformity to the axioms of the von Neumann no-go theorem on hidden variable theories of quantum mechanics which rules it out.

3. Spinors and quaternions.

Eli Cartan's *The theory of spinors,* contains a section which we use to indicate a mapping between the quaternion algebra of the Irish mathematician William Rowan Hamilton, and that of the spinors describing the relativistic (Dirac) equation of the electron.

Cartan introduces spinors via

$$x_1^2 + x_2^2 + x_3^2 = 0$$

and considers the matrices

$$H_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so that

$$H_1^2 = H_2^2 = H_3^2 = 1, H_1H_2 = -H_2H_1, H_2H_3 = -H_3H_2, H_3H_1 = -H_1H_3,$$

which he states are related to the quaternions, which form a division algebra, by

$$I_1 = -iH_1, I_2 = -iH_2, I_3 = -iH_3,$$

from which we deduce

$$I_1^2 = I_2^2 = I_3^2 = -1, I_1I_2 = -I_2I_1, I_2I_3 = -I_3I_2, I_3I_1 = -I_1I_3,$$

and he relates these to the Dirac equation. □

We state that locally we prefer the quaternion representation, and we will go into the reasons later. In order to retain the algebra of spinors, we introduce the idea that, say for the electron, the leptonic interior is globally twisted, so that a rotation of 2π radians of a quaternionic vector in a plane rotates the other two vectors by π radians with respect to the plane, and its localisation is immersed in an oriented quaternionic manifold. This lepton is a type of Möbius strip, but with 3 space componemts. In other words, we interpret the global oriented manifold as being the carrier of gravitation, so that the lepton-graviton interaction is locally oriented.

Fermi-Dirac statistics are sufficient to prevent a collapsed universe, although these collapsed states might coexist with other states. A Mohapatra algebra where the rotation by π radians is replaced by $2\pi/n$ is possible, but outside of the cases 2π (spin 1) and π (spin ½), which are the only trajectories which do not self-intersect, has not been observed in practice. Of course, we can have multiples of these spins ½ or 1.

I want to say something about why we should choose a twisted quaternionic (or novanionic) description for the electron, rather than the conventional Dirac equation, which is a spinor.

To form a division algebra, as we have mentioned already, and will do so further, is a desirable property. The twisted quaternion is locally embedded within a flat (gravitational) quaternionic manifold (cries of despair that we are not introducing general relativity here – I hope to discuss gravitation theories elsewhere, not in this paper). It is a good question as to whether a 3-dimensional space part of a twisted quaternion is feasible – that is, can be globally extended everywhere in the 3-dimensional part of the manifold. This may be related to what is known as the Bruhat-Tits construction. We can see that doing a circuit twice brings us back to the same orientation, so doing the circuit at right angles can do the same, and combining one twist circuits at right angles brings us to a point where two orientations at right angles are reversed – this seems OK, but we need to prove it properly. Now a Möbius strip cannot be globally embedded within a flat part of a manifold in which it is immersed, but locally it can. A feature of Möbius strips is that eventually, they are not contractible in the manifold in which they are immersed, and this is a feature of the world we live in – it is extended, and is not contracted to a point. Thus, unlike for spinors where we introduce Fermi-Dirac statistics as an afterthought, for twisted quaternionic manifolds we have a ready-made description of why a separation property is present.

4. Quaternions as the carriers of special relativity and the wave function.

The quaternionic inverse of the quaternion

$$a1 + bi + cj + dk,$$

so that their product is 1, is the quaternion

$$(a1 – bi – cj – dk)/(a^2 + b^2 + c^2 + d^2).$$

The scalar number $(a^2 + b^2 + c^2 + d^2)$ is known as the norm of the quaternion.

Quaternion multiplication is used in computing the motion of gyroscopes, where the i, j and k imaginary (quaternionic) numbers represent the three space directions.

We will take time as being real and space as being quaternionic imaginary. The special relativistic line element is given by

$$s^2 = c^2t^2 - x'^2 - y'^2 - z'^2.$$

We will represent this in terms of quaternions by use of the space coordinates denoted by

xi, yj and zk

as the quaternionic norm

$$s^2 = c^2t^2 - (xi)^2 - (yj)^2 - (zk)^2. \qquad (1)$$

We can then obtain the *Lorentz transformations* for special relativity using this idea, that is, expressions for ct and xi etc. in terms of the quaternionic velocity vi = d(xi)/dt. These are

$$t' = [t + \frac{vi}{c^2}xi]/\sqrt{1 + \frac{(vi)^2}{c^2}},$$

$$= [t - \frac{v}{c^2}x]/\sqrt{1 - \frac{v^2}{c^2}} \qquad (2)$$

and

$$x'i = [x - vt]i/\sqrt{1 + \frac{(vi)^2}{c^2}},$$

$$x' = [x - vt]/\sqrt{1 - \frac{v^2}{c^2}}, \qquad (3)$$

the quaternionic and Lorentz equations being *identical*, not similar. In the total dimensional case, substitute xi with xi + yj + zk; its square is $-(x^2 + y^2 + z^2)$.

As usually described, a timelike attribution of the 'probability of the wave function', although we are not using this interpretation, is similar to and is approximated by

$$|\Psi(ct, x''i, y''j, z''k)|^2 = c^2t^2 + x''^2 + y''^2 + z''^2,$$

where the quaternion wave function Ψ is multiplied by its quaternionic conjugate Ψ*:

$$|\Psi|^2 = \Psi\Psi^* = (ct + x''i + y''j + z''k)(ct - x''i - y''j - z''k) = c^2t^2 + x''^2 + y''^2 + z''^2.$$

Ψ is interpreted as a probability amplitude, whereas the norm or modulus ΨΨ* of the wave function is the probability density that the particle is at (ct, x″i, y″j, z″k), rather than elsewhere.

We indicate a possible error in the reasoning here: why not represent the wave function in the form (1) using xi, yj and zk, presumably under which all measurements take place? Rather than be involved in academic debate, we would prefer to subject this idea to experiment. Note that rescaling of probabilities to take negative values is perfectly feasible. We will develop this later.

Interaction physics is now described in terms of quaternionic multiplication of spaces (for which we will explore the possibility that these correspond to the de Broglie Bohm pilot wave), with addition of these spaces being linear, as for the wave function.

5. Division and novanion algebras, t = 0 and nonconservation of number.

Quaternions are a division algebra. This means that no two non-zero quaternions multiplied together equate to zero. The proof is given in our work on n-novanions: '*Superexponential algebra*', chapter V, and at the end of this section.

When a quaternion is zero, using it to divide another quaternion by zero is described using the zero algebras of our work '*Discussion of ladder numbers and zero algebras*' [Ad14b]. In particular, the non-conservation of number is a possibility, given by an indeterminate zero algebra.

Since we are equating the zero quaternion to the state at zero time with zero distance, we will allocate this to the beginning of the universe, in which number is created, under allowable multiplicative interaction of quaternionic quantum states.

The more complicated situation for novanions is an extended space theory at t = 0, in which the quaternions can be positioned at zero space.

These ideas can be grafted on to various universe creation theories currently circulating, and without commitment we sketch them.

Embedded in Hawking-Turok Instanton theory, this explains the existence of the instanton in the first place**.**

In a Protouniverse theory, developed to explain the non-uniformity and the varying density of the universe, the formation of matter from nothingness before the big bang is related to white hole theory, where matter continuously appears at the speed of light, but there is believed to be no observational evidence for white holes.

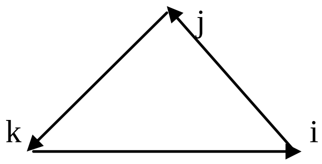
Alan Guth initiated Inflationary theory in 1981. Guth, an expert in scalar fields explaining how elementary particles got their mass, combined the mathematical equations for a scalar field with Einstein's equations describing the expansion of the universe, and developed a theory in which large amounts of matter and energy were created from nothing. After matter and energy were created, the universe experienced an accelerated expansion, becoming exponentially large prior to continuing its evolution according to the big bang model. This theory has been worked on and modified by many cosmologists since its introduction.

In Andre Linde's Self Creating Universe, the bubble universe involves creation of universes from a quantum foam of parent universes. On very small scales, the foam is frothing due to energy fluctuations. These fluctuations may create tiny bubbles and wormholes. If the energy fluctuation is not very large, a tiny bubble universe may form, experience some expansion like an inflating balloon, and then contract and disappear from existence. However, if the energy fluctuation is greater than a particular critical value, a tiny bubble universe forms from the parent universe, experiences long-term expansion, and allows matter and large-scale galactic structures to form. The theory stems from the concept that each bubble or inflationary universe will sprout other bubble universes, which in turn, sprout more bubble universes. The universe we live in has a set of physical constants that seem tailor-made for the evolution of living things. On the other hand, our theory stipulates the existence of only one t = 0, and so does not correspond to this Self Creating Universe theory.

Note that we are not adopting here the Einsteinian interpretation of the line element, in which special relativity is described by (here quaternionic or novanionic) vectors without base point. We allocate a base point to all quaternionic and novanionic vectors, so that they do not 'float', which is the pre-Einsteininan idea due to Lorentz. Then there is a specific and unique zero time and zero space.

The existence of simultaneity within this description is now available. This is especially useful in general relativity if adopted, where the manifold can globalise in other Einsteinian circumstances in multiple ways from its many localisations. The presence of simultaneity guarantees the existence of a unique globalized manifold.

The quaternions may be represented by a triangle diagram



where the direction of the arrows indicates the sign in products and the nodes i, j and k satisfy

$$ij = k,\ jk = i,\ ki = j, \quad (1)$$

in other words, for a positive sign in the above relations, we are following the arrows. When we are going in a direction opposite to the arrows, we have a negative sign:

$$ji = -k,\ kj = -i,\ ik = -j. \quad (2)$$

We have here that 1 commutes with all elements, and also

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1. \quad (3)$$

Some matrix representations of the quaternions are given in [Ad14b], chapter 6. From that work we can describe the hyperintricate representation of quaternions.

To demonstrate that the product of two quaternions with real coefficients

$$(a1 + bi + cj + dk)(p1 + qi + rj + tk) \quad (4)$$

cannot be zero unless a = b = c = d = 0 or p = q = r = t = 0, we will adopt a proof ‘the long way round’, which we will simplify and extend later to the nonassociative 10-novanions.

Equating by hypothesis the real and quaternionic parts of (4) to zero, we obtain the following 4 equations in 8 unknowns.

<u>real part</u>:

$$ap – bq – cr – dt = 0 \quad (5)$$

<u>i part</u>:

$$pb + aq + ct – dr = 0 \quad (6)$$

<u>j part</u>:

$$pc + ar – bt + qd = 0 \quad (7)$$

<u>k part</u>:

$$pd + at + br – cq = 0. \quad (8)$$

If a = 0, then equations (6), (7) (8) and (9) imply

$$(b^2 + c^2 + d^2)q = 0, \quad (9)$$

$$(b^2 + c^2 + d^2)r = 0 \quad (10)$$

and

$$(b^2 + c^2 + d^2)t = 0, \quad (11)$$

so either b = c = d = 0, which we exclude, or q, r and t = 0, which implies (4) cannot be satisfied except for b = c = d = 0.

On eliminating p, provided a ≠ 0 we obtain

$$(a^2 + b^2)q + (-ad + cb)r + (ac + db)t = 0, \quad (12)$$

$$(ad + cb)q + (a^2 + c^2)r + (-ab + cd)t = 0 \quad (13)$$

and

$$(-ac + bd)q + (ab + cd)r + (a^2 + d^2)t = 0. \quad (14)$$

On eliminating q,

$$[(a^2 + c^2)(a^2 + b^2) + (ad + cb)(ad - cb)]r + [(-ab + cd)(a^2 + b^2) - (ad + cb)(ac + db)]t = 0 \quad (15)$$

and

$$[(ab + cd)(a^2 + b^2) - (-ac + db)(-ad + cb)]r + [(a^2 + d^2)(a^2 + b^2) - (-ac + db)(ac + db)]t = 0. \quad (16)$$

Then on eliminating r, whilst putting

$$A = a^2, B = b^2, C = c^2, D = d^2, \quad (17)$$

we obtain provided p = q = r = t = 0 does not hold

$$\begin{aligned}&[(A + C)(A + B) + (AD - CB)][(A + D)(A + B) - (-AC + DB)]\\ &\quad - [(A + B)^2(-AB + CD) + (-AD + CB)(-AC + DB)]\\ &\quad + (-ab + cd)(A + B)(-ac + db)(-ad + cb) + (ab + cd)(A + B)(ad + cb)(ac + db)\\ &\qquad = 0. \end{aligned} \quad (18)$$

The last line with terms in (18) is

$$2(A + B)[BCD + ACD + ABD + ABC], \quad (19)$$

so we have

$$\begin{aligned}&(A + B)^2A[A + B + C + D]\\ &\quad + (A + B)[A^2D + AD^2 + A^2C + AC^2 + ABC + ACD + ABD] = 0 \end{aligned} \quad (20)$$

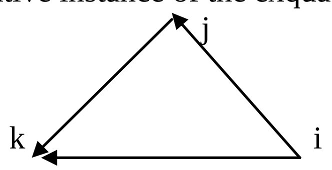
which is impossible, since $A > 0$. □

6. Exquaternions, and chirality of electroweak interactions.

We have introduced exquaternions in chapter 6 of '*Innovation in mathematics*' [Ad14b] and our work on n-novanions [Ad14a] where the nonassociative exquaternions (which therefore cannot be represented by a matrix) can be described by the multiplication table on elements

×	***1***	***i***	***j***	***k***
1	1	i	j	k
i	i	–1	k	j
j	j	–k	–1	i
k	k	–j	–i	–1

A representative instance of the exquaternions can be depicted in the diagram



where the direction of the arrows indicates the sign in products. For this particular example

$$(j - k)(1 - i) = j + k - k - j = 0, \quad (1)$$

so that the exquaternions do not form a division algebra. □

Equaternions have a different handedness – in physics the term often used is that they have different chirality – than the quaternions.

Our interpretation of these results is that in the early universe exquaternions were present, but since interaction is described by quaternionic multiplication, they self-annihilated, that is, they interacted by multiplication to produce the zero state. This means that the resulting handedness of the universe is fixed (left-handed), as is shown in the electroweak interactions, for instance in the handedness of β decay.

7. Octonions as division algebras.

We provide a proof that division algebras with more than 4 basis elements are nonassociative. For a basis element A whose square is 1, the inverse $A^{-1} = A$. For any basis element, B, whose square is -1, the inverse $B^{-1} = -B$. An associative matrix algebra may be completely represented in terms of the hyperintricate representation of matrices documented in [Ad14b].

There are only two possibilities for these basis elements, they either commute, AB = BA, or they anticommute, AB = -BA.

Consider finding the inverse of $aA_1 + bA_2$, $A_1 \neq A_2$, where A_1^2 and $A_2^2 = 1$. This is then

$$(aA_1 - bA_2)/(a^2 - b^2) \qquad (1)$$

when A_1 and A_2 commute and

$$(aA_1 + bA_2)/(a^2 + b^2)$$

when A_1 and A_2 anticommute. If we incorporate the fact that 1 is always present amongst such A's, then for some values of a and b, (1) holds, which implies that there exist a's and b's for which (1) includes the possibility of dividing by zero. The statement that we can do division is incorporated in the definition of a division algebra (although we have to specifically exclude division by zero, as for a field), therefore there exists in such division algebras only one basis element with square 1, and this must be the real basis element.

To find the inverse of $a1 + bB_1$, where $B_1^2 = -1$, then this is

$$(a1 - bB_1)/(a^2 + b^2),$$

which introduces no further problems.

To find the inverse of $aB_1 + bB_2$, for $B_1^2 = -1$ and $B_2^2 = -1$, then this is the permissible

$$-(aB_1 + bB_1)/(a^2 + b^2),$$

when B_1 and B_2 anticommute, which is now the only possibility.

The above argument may be generalised for more B_r's, and it becomes necessary to stipulate that all $B_1, B_2, \ldots B_n$ mutually anticommute.

We know there are solutions for B_1, B_2, B_3 given by basis elements for the quaternions. Now assume the existence of four such basis elements, B_1, B_2, B_3, B_4, all mutually anticommuting and distinct, so that $B_rB_s = -B_sB_r$. We will use associativity of these basis elements in computing from $B_1B_2B_3B_4$ its mirror reflection in two separate ways. So

$$\begin{aligned} B_1B_2B_3B_4 &= -B_1B_2B_4B_3 \\ &= B_1B_4B_2B_3 \\ &= -B_4B_1B_2B_3 \\ &= B_4B_1B_3B_2 \\ &= -B_4B_3B_1B_2 \\ &= B_4B_3B_2B_1. \end{aligned}$$

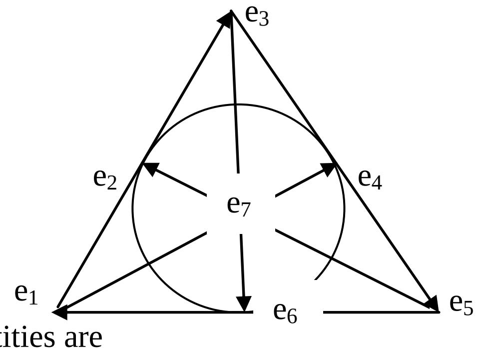
However

$$\begin{aligned} (B_1B_2)(B_3B_4) &= -(B_3B_4)(B_1B_2) \\ &= -(B_4B_3)(B_2B_1), \end{aligned}$$

a contradiction.

Thus the maximum number of dimensions for an associative division algebra is 4. ☐

The 8-dimensional nonassociative octonions, with one real component and seven octonionic imaginary components are described by a diagram known as the Fano plane.



Typical cyclic identities are

$e_7e_5 = e_2$ (2)

$e_5e_6 = e_1$

and

$e_4e_2 = e_6$.

Note what we have said here. The inner triple $e_2e_4e_6$ acts like a quaternion, but the outer triple $e_1e_3e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple = $e_2e_4e_6$ + central triple = $e_1e_3e_5$ + one = e_7. Octonions form a division algebra, in particular

$e_c^2 = -1$, (3)

$e_ae_b = -e_be_a$ $(a \neq b)$,

and the inverse of

$a1 + \sum_{n=1}^{7} b_n e_n$

is

$(a1 - \sum_{n=1}^{7} b_n e_n)/(a^2 + \sum_{n=1}^{7} b_n^2)$. (4)

The octonions, $\mathbb{O}$, are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a ∗-algebra to be an algebra equipped with conjugation, a linear map * satisfying

$a^{**} = a$, (5)

$(ab)^* = b^*a^*$. (6)

Starting from any ∗-algebra, the Cayley-Dickson construction gives a new algebra

$(a, b)(c, d) = (ac - db^*, ad^* + cb)$, (7)

with conjugation defined by

$(a, b)^* = (a^*, -b)$. (8)

This generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$ of the octonions

×	***1***	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We can generate for each

$\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$: $e_i \times e_j \rightarrow e_k$

a Cayley-Dickson construction of

$e_i \times (-e_j) \rightarrow -e_k$,

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$$

Thus distinct possibilities for the Fano plane are isomorphic to distinct instances of the Cayley-Dickson construction. □

A Fano plane can be devised to form the exoctonions. Representing the exquaternions of section 2 by the 4 × 4 block A, the exoctonion multiplication table may be represented by the blocks

A B
C D

where A or its transpose A^T may be chosen and D or its transpose D^T. An example of the complex numbers, exquaternions and exoctonions is represented in the following nested table for an instance of the exoctonions.

×	***1***	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	–1	e_3	e_2	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	–1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	$-e_2$	$-e_1$	–1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	–1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	–1	e_3	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	$-e_3$	–1	e_1
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	–1

The exoctonions do not form a division algebra, since they contain the exquaternions as a subalgebra, but the inverse of $a_0 1 + a_1 e_1 + a_2 e_2 + \ldots$ etc. exists and is

$$(a_0 1 - a_1 e_1 - a_2 e_2 - \ldots \text{ etc.})/(a_0^2 + a_1^2 + a_2^2 + \ldots \text{ etc.}).$$

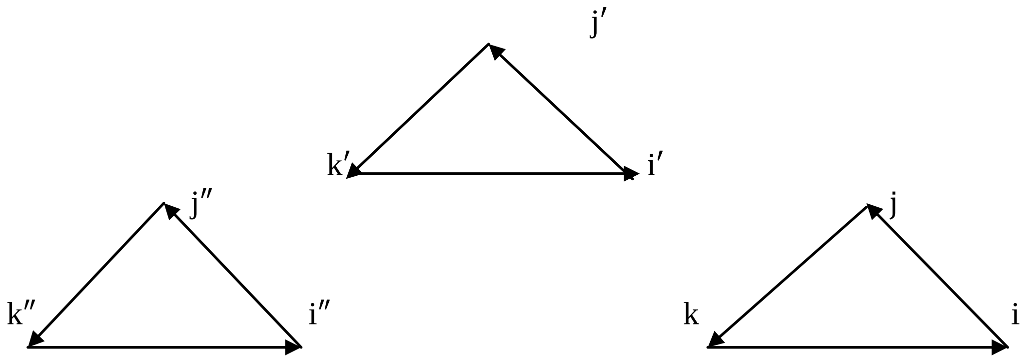
There is a connection between the algebra of octonions and the exceptional Lie algebras.

Our original intention was to use the octonions, which have an eight dimensional norm, to extend the number of relativistic dimensions, enabling a description of an embedded general relativity and further electroweak states. Following the suggestion of a first-year physics student, we now have a better idea.

8. The 10-novanions.

We introduce the 10-novanions. This is lifted from the n-novanion paper, including the proof that they form a novanion algebra (see the work of the mathematician J. F. Adams *On the non existence of elements of Hopf invariant one* for the more limiting proof for division algebras).

The 10-novanions are represented by the set of triangle diagrams where in general each triangle is a quaternion without 1. They have a norm of the usual type.



The primed variables ( ), (′) and (″) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k′) and (k″), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = -\,k'k. \tag{1}$$

When the variables contain different instances, such as k and i′, then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \tag{2}$$

and consequently

$$ki' = j'' = -i'k. \tag{3}$$

Our claim is that the inverse of

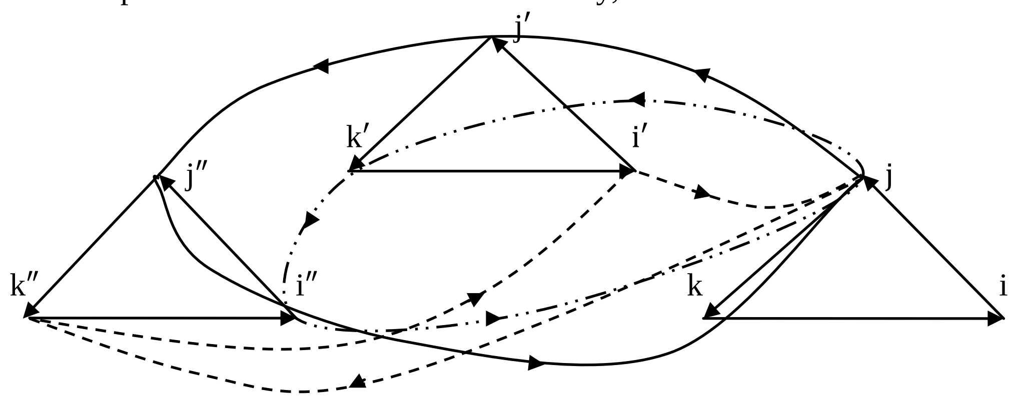
$$a1 + \sum_{n=1}^{3}\sum_{\text{primed } m=1}^{3} b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^{3}\sum_{\text{primed } m=1}^{3} b_n^m e_n))/(a^2 + (\sum_{n=1}^{3}\sum_{\text{primed } m=1}^{3}(b_n^m)^2)), \tag{4}$$

and this constitutes a division algebra with no divisors of zero – the 10 dimensional 10-novanions.

In order to picture the 10-novanions more closely, we will show the connections from node j



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and this constitutes a type of division algebra with no divisors of zero provided $a1 \neq 0$ – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times (p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \quad (5)$$

Their product is

<u>real part</u>:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \quad (6)$$

<u>i part</u>:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \quad (7)$$

<u>j part</u>:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \quad (8)$$

<u>k part</u>:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \quad (9)$$

<u>i′ part</u>:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \quad (10)$$

<u>j′ part</u>:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \quad (11)$$

<u>k′ part</u>:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \quad (12)$$

<u>i″ part</u>:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \quad (13)$$

<u>j″ part</u>:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \quad (14)$$

<u>k″ part</u>:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \quad (15)$$

If $a = 0$, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying $a = 0$ do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \quad (16)$$

From now on we will assume $a \neq 0$. By a symmetrical argument applied also to the theorem which follows, we need to assume with this that $p \neq 0$.

Equations (6) to (15) form a matrix $E + aI$, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector $(p, q, r, t, q', r', t', q'', r'', t'')$. Below we give a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, provided in chapter 11 of [Uh01], so these correspond to $-a$, which is real, whereas we are now excluding the only possibility for this, $a = 0$. □

Within the field of complex numbers ℂ, the complex conjugate of $c = a + ib$ is $c^* = a - ib$. For the corresponding matrix C with entries $c_{jk} = a_{jk} + ib_{jk}$, the conjugate $C^* = a_{jk} - ib_{jk}$. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^TC^T.$$

A matrix is defined as antisymmetric if $C^T = -C$.

Theorem: *All eigenvalues of a real antisymmetric matrix* $E = -E^T$ *are pure imaginary*.

Proof. Consider the case of the eigenvalue λ and possibly complex eigenvector $\mathbf{x} \neq \mathbf{0}$. According to the Fundamental Theorem of Algebra $\lambda \in \mathbb{C}$. Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \tag{17}$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (17), we obtain

$$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*.$$

Transposing yields

$$(E\mathbf{x})^{*T} = x^{*T}E^{*T} = \lambda^*\mathbf{x}^{*T}.$$

Define the norm

$$||E\mathbf{x}||^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \tag{18}$$

where

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since $E^T = -E$ and $E^{*T} = E^T$ for a real antisymmetric matrix E, we can write (18) as

$$\begin{aligned} ||E\mathbf{x}||^2 &= \mathbf{x}^{*T}E^{*T}\, E^T\mathbf{x}, \\ &= \mathbf{x}^TE^TE\mathbf{x} \\ &= -\mathbf{x}^TE^2\mathbf{x} \\ &= -\mathbf{x}^T\lambda^2\mathbf{x}, \end{aligned}$$

because $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$||E\mathbf{x}||^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \tag{19}$$

Now $\mathbf{x}^T\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (18) and (19). Thus a real antisymmetric matrix $E = -E^T$ can only have imaginary eigenvalues λ.

Thus, the 10-novanions form a novanion algebra. ☐

9. Interpretation of 10-novanions as the carrier of the 10-dimensional heterotic string.

References are Witten, Schwarz and Green, and Polyakov. There exists within string theory the basic examples of the 10-dimensional heterotic string and the 26-dimensional bosonic string.

We now have the identification of 10-novanions with the 10-dimensional heterotic string.

According to what we have already presented, this is anomaly-free, relativistic and obeys the relativistic Dirac equation in its three subalgebras (no supersymmetry is needed).

A feature of pure quaternions is that they are readily identifiable with the 3-space and 1-time we live in. For 10-novanions, it is interesting to consider each subquaternion in this geometric way.

10. Interpretation of 10-novanions as neutrinos.

The 10-novanions encompass 3 families of quaternions, and we raise the question as to whether these can be identified with the three families of leptons.

Further, these exist not in a single state, but there is the possibility of mixing between these states, as is available in the evidence on solar neutrino flux.

In the Standard Model each lepton starts out with no intrinsic mass. The charged leptons (that is, the electron, muon, and tau) obtain an effective mass through interaction with the Higgs field, but the neutrinos remain massless. For technical reasons the masslessness of the neutrinos implies that there is no mixing of the different generations of charged leptons as there is for quarks. This is in close agreement with current experimental observations. [PS95]

However, it is known from experiments – most prominently from observed neutrino oscillations [Fu98] – that neutrinos do in fact have some very small mass, probably less than 2 eV/c2 [Am08]. This implies the existence of physics beyond the Standard Model. The currently most favoured extension is the so-called seesaw mechanism, which explains why the left-handed neutrinos are so light compared to the corresponding charged leptons.

However, neutrino oscillations are known to violate the conservation of the individual leptonic numbers. Such a violation is considered to be smoking gun evidence for physics beyond the Standard Model. A much stronger conservation law is the conservation of the total number of leptons (L), conserved even in the case of neutrino oscillations, but even it is still violated by a tiny amount by the chiral anomaly.

11. The n-novanions.

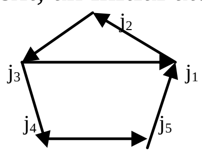
Again, this will be lifted from the n-novanion paper. The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ) , (′) and (″), and by a directly analogous procedure this constitutes a $1 + 3 \times 7 = 22$ dimensional division algebra.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say (′), (′)′ and (′)″, we obtain in general an $n = 1 + 3^f7^g$ dimensional novanion algebra, the n-novanions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k′)′ and (k″)″ is evaluated. □

Note that, just as we introduced the exquaternions, in a perfectly analogous fashion we can assemble exnovanions. □

12. The bosonic string.

Are there other division algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student’s identification of novanions with strings in physics, [Ad14b], for which we wish to investigate the bosonic allocation $1 + 5^2$. In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



the pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3,\ j_2j_3 = j_4,\ j_3j_4 = j_5,\ j_4j_5 = j_1,\ j_5j_1 = j_2, \qquad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2,\ j_4j_2 = j_3,\ j_5j_3 = j_4,\ j_1j_4 = j_5,\ j_2j_5 = j_1, \qquad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \sum_{n=1}^{5} b_n j_n$$

is

$$a1 - \sum_{n=1}^{5} b_n j_n/(a^2 + \sum_{n=1}^{5} b_n^2), \qquad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3 j_1)j_4 = -j_3 \neq j_3(j_1 j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f 5^g 7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \qquad (4)$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

<u>real part</u>:

$$ap - bq - cr - dt - eu - fv = 0, \qquad (5)$$

<u>j_1 part</u>:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \qquad (6)$$

<u>j_2 part</u>:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \qquad (7)$$

<u>j_3 part</u>:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \qquad (8)$$

<u>j_4 part</u>:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \qquad (9)$$

<u>j_5 part</u>:

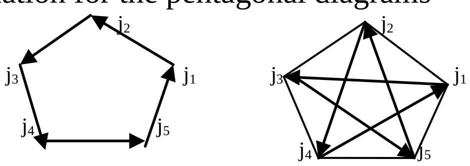
$$fp - eq + 0 - et + (d + b)u + av = 0, \qquad (10)$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$. □

We find a similar type of situation for the pentagonal diagrams



and we have failed by introducing modifications of equations (5) to (10) to come up with other solutions, where for pentagonal diagrams an exhaustive analysis shows these do not exist. □

However, the identification relates to the number 1 + 25, and we now wish to probe the allocation 25 = 7 + 9 + 9, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as 1 + 3 + 3, where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

$3_a, 3_b, 3_c$ (i)

$3_d, 3_e, 3_f$ (ii)

$3_g, 3_h, 1_u$ (iii)

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word 'similar' later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from 3_g, 3_e to 3_c. This is similar to a 10-novanion algebra. The descending line from 3_a, 3_e to 1_u is an octonion algebra. We then incorporate the algebra taking for example 3_d, 3_b to 1_u, an octonion algebra, or 3_g, 3_b to 3_f, this is similar to a 10-novanion algebra.

We have used the words 'similar to a 10-novanion algebra', and we now explain why. If we look at allocation (iii), this is part of the 3_g 3_h 1_u octonion, where 3_h and 1_u are linked. Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by 3_a 3_d 3_g is a 10-novanion, since it is made of genuine quaternions, but the vertical allocation 3_b 3_e 3_h is not. 3_b, 3_e and 3_h occur in octonion representations. If we were to state that the central triple 3_b, 3_e and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a 'similar to 10-novanion' structure, we decide that the octonion structure overrides the 10-novanion one. Since there is only one special 1_u part for the octonions, this allocation is unique. The similar 10-novanion structure is now not a closed algebra within the 10-novanions; part of it belongs to the octonions.

Since there is no other mixing of allocations, the result is as consistent as the 10-novanions and the octonions. This can be checked with equations like 8.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 8.(16) shows that this is a novanion algebra. □

The existence of 10-, 26- and 80-novanions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a $(3^m \pm 1)$-novanion) implies that results derived for division algebras have a different extension for n-novanions. □

13. Sedenions and 64-novanions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all x and y in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

For the example table that follows

$$(e_1 + e_{10})(e_{15} - e_4) = 0. \square$$

The standard complex numbers, quaternions, octonions and sedenions can be nested by inclusion and an instance is represented in the following table for the sedenions

×	***1***	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative in 1960. If we introduced the following diagram

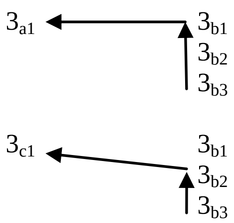
$$\begin{array}{ccc} 3_a, 3_b, 1_s & & \\ & 1_t & \\ 3_c, 3_d, 1_u & & \end{array}$$

where octonions 3_a, 3_b give $\pm 1_t$ and 3_c, 3_d also gives $\pm 1_t$, allocation 1_s, 1_t 1_u is a quaternion, then we have for octonions 3_a 3_c gives $\pm 1_s$; 3_b, 3_d gives $\pm 1_s$, then 3_c and 3_b is matched with $\pm 1_u$ and 3_a and 3_d with $\pm 1_u$, then this configuration is inconsistent in all configurations.

Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novanion algebras of this type are consistent. A 64-novanion is given by the cube with slices

$$\begin{array}{lll} 3_a, 3_b, 1_p & 3'_a, 3'_b, 1'_p & 3''_a, 3''_b, 1''_p \\ 3_c, 3_d, 1_q & 3'_c, 3'_d, 1'_q & 3''_c, 3''_d, 1''_q \\ 3_e, 3_f, 1_r & 3'_e, 3'_f, 1'_r & 3''_e, 3''_f, 1''_r. \end{array}$$

To evaluate a typical slice algebra, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b. We will need to look at this typical example in detail, so denote the three elements of 3_b by 3_{b1}, 3_{b2} and 3_{b3}. We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. Of course, two arrows combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_a, 3_c and 3_e. The 1_p, 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures (3_a, 3_b, 1_p), (3_c, 3_b, 1_r) and (3_e, 3_b, 1_q). Because 3_{b1} links to 3_{a1}, we have to ensure that the link 1_q to 3_{b1} does not also link to 3_{a1}, but this can be arranged. □

This conclusion is linked to questions similar to those about division algebras in homotopy theory, and we will investigate connections with the following circumstances

(i) The homotopy group for a sphere.
(ii) Steenrod square operations.
(iii) Parallelisation of vector fields on spheres.
(iv) Fiber bundle structures.
(v) K(X) as a universal description for objects described by semigroups.
(vi) Bott periodicity.
(vii) Universal Quillen homotopy. □

Since the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are dependent on the existence of division algebras limited in number to those equivalent to the octonions, [Wi09]. An investigation of issues (i) to (vii) above and the implications of the existence of algebras of novanion type for the classification of Lie algebras and of groups will be given in *Branched spaces* [Ad16].

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